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1. Find the fixed points of each complex function and classify them as attracting, repelling, or neutral:

   (a) $f(z) = z + z^2(z-i)(z+1)$.

   - The fixed points satisfy $z = z + z^2(z-i)(z+1)$, so $z^2(z-i)(z+1) = 0$, with roots $z = \boxed{0,\ i,\ -1}$.
   - We have $f'(z) = 1 + 2z(z-i)(z+1) + z^2(z+1) + z^2(z-i)$, so $f'(0) = 1$, $f'(i) = 1 + i^2(i+1) = -i$, and $f'(-1) = 1 + (-1)^2(-1-i) = -i$.
   - Then $|f'(0)| = |f'(i)| = |f'(-1)| = 1$, so all three points are $\boxed{\text{neutral}}$.

   (b) $f(z) = z^2 - 3z + 5$.

   - The fixed points satisfy $z = z^2 - 3z + 5$, so $z^2 - 4z + 5 = 0$, with roots $z = \dfrac{4 \pm \sqrt{-4}}{2} = \boxed{2+i,\ 2-i}$.
   - We have $f'(z) = 2z - 3$, so $|f'(2+i)| = |1+2i| = \sqrt{5}$ and $|f'(2-i)| = |1-2i| = \sqrt{5}$.
   - Thus, both points are $\boxed{\text{repelling}}$.

   (c) $f(z) = z^2 - 2iz + (i-1)$.

   - The fixed points satisfy $z = z^2 - 2iz + (i-1)$, so $z^2 - (1+2i)z + (-1+i) = 0$, with roots $z = \dfrac{(1+2i) \pm \sqrt{(1+2i)^2 - 4(-1+i)}}{2} = \dfrac{(1+2i) \pm 1}{2} = \boxed{1+i,\ i}$.
   - We have $f'(z) = 2z - 2i$, so $|f'(1+i)| = |2| = 2$ and $|f'(i)| = |2i - 2i| = 0$.
   - Thus, $\boxed{1+i \text{ is repelling}}$ while $\boxed{i \text{ is attracting}}$.

   (d) $f(z) = iz^2 + z + i/4$.

   - The fixed points satisfy $z = iz^2 + z + i/4$, so $i(z^2 + 1/4) = 0$, with roots $z = \boxed{i/2,\ -i/2}$.
   - We have $f'(z) = 2iz + 1$, so $|f'(i/2)| = |0| = 0$ and $|f'(-i/2)| = |2| = 2$.
   - Thus, $\boxed{i \text{ is attracting}}$ while $\boxed{-i \text{ is repelling}}$.

   (e) $f(z) = 2z^3 + 2z$.

   - The fixed points satisfy $2z^3 + 2z = z$ so $z(2z^2 + 1) = 0$ with roots $z = \boxed{0, \pm i\sqrt{2}/2}$.
   - We have $f'(z) = 6z^2 + 2$, so $|f'(0)| = |2| = 2$ and $\left|f'(\pm i\sqrt{2}/2)\right| = |-3+2| = 1$.
   - Thus, $\boxed{0 \text{ is repelling}}$ while $\boxed{\pm i\sqrt{2}/2 \text{ are neutral}}$.

   (f) $f(z) = 1 + \dfrac{4i}{z + 2 - 3i}$.

   - The fixed points satisfy $z = \dfrac{z+2+i}{z+2-3i}$, so $z^2 + (2-3i)z = z + 2 + i$, or $z^2 + (1-3i)z + (-2-i) = 0$.
   - Solving this quadratic equation yields $z = \dfrac{(-1+3i) \pm \sqrt{-2i}}{2} = \dfrac{(-1+3i) \pm (1-i)}{2} = \boxed{i,\ -1+2i}$.
   - We have $f'(z) = -\dfrac{4i}{(z+2-3i)^2}$, so $|f'(i)| = \left|\dfrac{-4i}{(2-4i)^2}\right| = \left|\dfrac{1}{2}\right| = \dfrac{1}{2}$, so $\boxed{i \text{ is attracting}}$.
   - Also, $|f'(-1+2i)| = \left|\dfrac{-4i}{(1-i)^2}\right| = |2| = 2$, so $\boxed{-1+2i \text{ is repelling}}$.

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2. For each function $f(z)$, the given value $z_0$ is a periodic point. Find its period and classify the associated cycle as attracting, repelling, or neutral:

   (a) $f(z) = 1 - i + iz$ with $z_0 = 2$.

   - We have $2 \to 1+i \to 0 \to 1-i \to 2$ so the period is 4.

- Since $f'(z) = i$ we have $f'(2) = f'(1+i) = f'(0) = f'(1-i) = i$ so by the chain rule formula we see that $\left|(f^4)'(2)\right| = |i \cdot i \cdot i \cdot i| = 1$. Since this quantity equals 1, the cycle is $\boxed{\text{neutral}}$.

(b) $f(z) = z^2 + i$ with $z_0 = -i$.

- We have $-i \to -1+i \to -i$ so the period is 2.
- Since $f'(z) = 2z$ we have $f'(-1+i) = -2+2i$ and $f'(-i) = -2i$, so by the chain rule formula we see that $\left|(f^2)'(-1+i)\right| = |(-2+2i)(2i)| = \sqrt{32}$. Since this quantity is greater than 1, the cycle is $\boxed{\text{repelling}}$.

(c) $f(z) = 1 - \dfrac{3}{2}z^2 - \dfrac{1}{2}z^3$ with $z_0 = 0$.

- We have $0 \to 1 \to -1 \to 0$ so the period is 3. Since $f'(z) = -3z - \dfrac{3}{2}z^2$ we see $f'(0) = 0$, $f'(1) = -\dfrac{9}{2}$, and $f'(-1) = \dfrac{3}{2}$, so by the chain rule formula we see that $\left|(f^3)'(0)\right| = \left|0 \cdot (-\dfrac{9}{2}) \cdot \dfrac{3}{2}\right| = 0$. Since this quantity is less than 1, the cycle is $\boxed{\text{attracting}}$.

(d) $f(z) = 3z + 4/z$ with $z_0 = i$.

- We have $i \to -i \to i$ so the period is 2. Since $f'(z) = 3 - \dfrac{4}{z^2}$ we see $f'(i) = f'(-i) = 7$ so by the chain rule formula we see that $\left|(f^2)'(i)\right| = |7 \cdot 7| = 49$. Since this quantity is greater than 1, the cycle is $\boxed{\text{repelling}}$.

(e) $f(z) = z^2$ with $z_0 = \cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9} = e^{2\pi i/9}$.

- We have $e^{2\pi i/9} \to e^{4\pi i/9} \to e^{8\pi i/9} \to e^{16\pi i/9} \to e^{32\pi i/9} = e^{14\pi i/9} \to e^{28\pi i/9} = e^{10\pi i/9} \to e^{20\pi i/9} = e^{2\pi i/9}$ so the period is 6. Since $f'(z) = 2z$ we see $f'(e^{i\theta}) = 2e^{i\theta}$ so by the chain rule formula we see that $\left|(f^6)'(e^{2\pi i/9})\right| = \left|2e^{2\pi i/9} \cdot 2e^{4\pi i/9} \cdot 2e^{8\pi i/9} \cdot 2e^{16\pi i/9} \cdot 2e^{14\pi i/9} \cdot 2e^{10\pi i/9}\right| = 64$. Since this quantity is greater than 1, the cycle is $\boxed{\text{repelling}}$.

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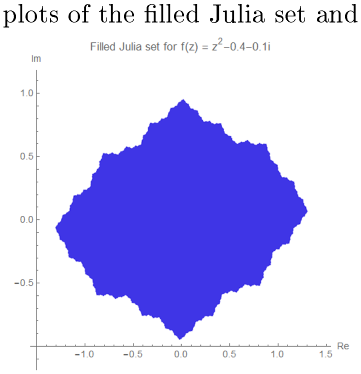
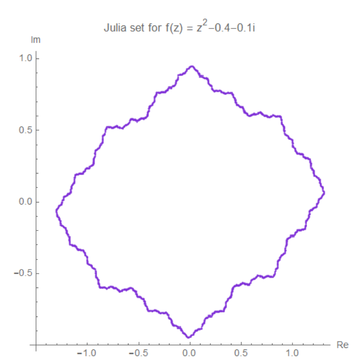
3. For the following quadratic functions, (i) plot the Julia set / filled Julia set, (ii) use the picture to identify whether the Julia set for that function is connected or disconnected, and then (iii) justify your answer by computing the orbit of the critical point and using the fundamental dichotomy.

(a) $f_1(z) = z^2 - 0.4 - 0.1i$.

- Here are the plots of the filled Julia set and Julia set:

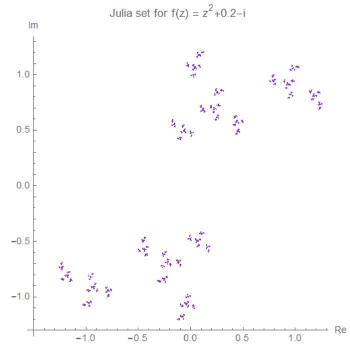



- The Julia set appears to be a closed curve enclosing a connected region (namely the filled Julia set).
- Numerically computing the critical orbit shows that it rapidly converges to the fixed point $z_0 \approx -0.3086 - 0.0618i$.
- Since the critical orbit stays bounded, by the fundamental dichotomy that the Julia set is $\boxed{\text{connected}}$.

(b) $f_2(z) = z^2 + 0.2 - i$.

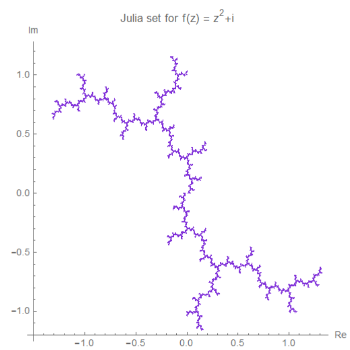
- Here is a plot of the Julia set:




- The Julia set appears to be a disconnected Cantor set.
- Numerically computing the first few terms of the critical orbit yields $\{0, 0.2-i, -0.76-1.4i, -1.1824+1.128i, 0.3257-3.6675i\}$. At this point, the orbit has escaped the circle $|z| = 2$ so by the escape criterion it will necessarily escape to $\infty$.
- Since the critical orbit escapes, by the fundamental dichotomy the Julia set is a $\boxed{\text{disconnected Cantor set}}$.

(c) $f_3(z) = z^2 + i$.

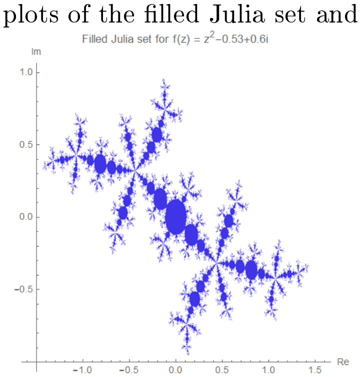
- Here is a plot of the Julia set:

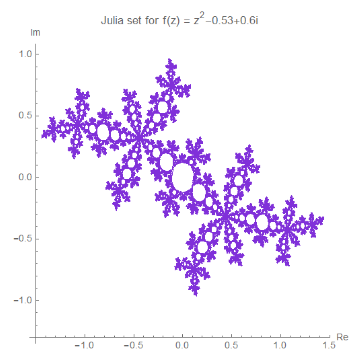



- The Julia set appears to be a connected "dendrite" set: it is not a curve and does not enclose a region, but it is connected.
- Computing the first few terms of the critical orbit yields $\{0, i, -1+i, -i, -1+i, -i, \dots\}$ so we see that 0 is eventually periodic and hence has a bounded orbit.
- Since the critical orbit stays bounded, by the fundamental dichotomy the Julia set is $\boxed{\text{connected}}$.

(d) $f_4(z) = z^2 - 0.53 + 0.6i$.

- Here are the plots of the filled Julia set and Julia set:



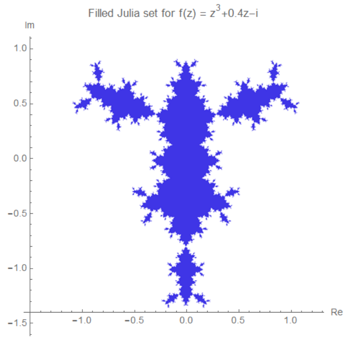



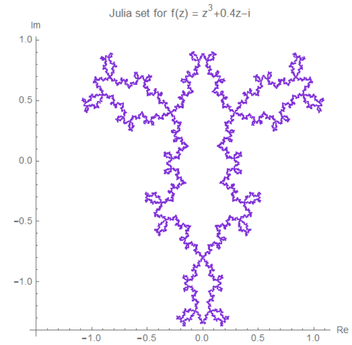

- The Julia set appears to be connected, but rather complicated. It does appear to enclose a number of connected regions making up the filled Julia set.
- Numerically computing 1000 iterates of the critical point shows that it converges to a 10-cycle, given approximately by $\{0.023-0.007i, -0.530+0.600i, -0.609-0.035i, -0.160+0.643i, -0.917+0.394i, 0.156-0.123i, -0.521+0.561i, -0.574+0.015i, -0.201+0.582i, -0.829+0.366i\}$.
- Since the critical orbit stays bounded, by the fundamental dichotomy the Julia set is $\boxed{\text{connected}}$.

4. For the following non-quadratic functions, (i) plot the Julia set and filled Julia set and (ii) use the picture to identify whether the Julia set for that function seems to be connected or disconnected.

   (a) $f_5(z) = z^3 + 0.4z - i$.

   - Here are the plots of the filled Julia set and Julia set:

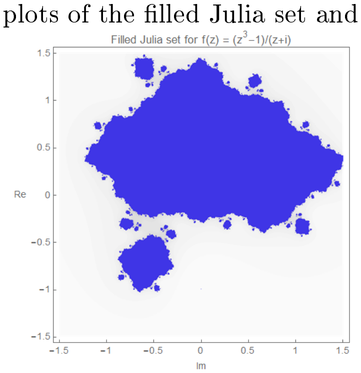


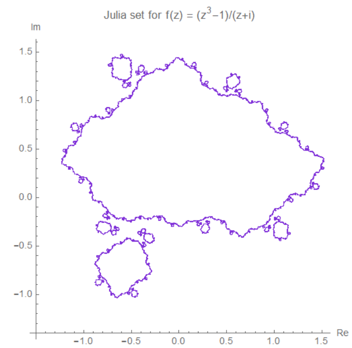




   - The Julia set appears to be $\boxed{\text{connected}}$, but rather complicated. It does appear to enclose a number of connected regions making up the filled Julia set (which bears a vague resemblance to a lobster, with two large claws and a tail).

   (b) $f_6(z) = \dfrac{z^3 - 1}{z + i}$.

   - Here are the plots of the filled Julia set and Julia set:

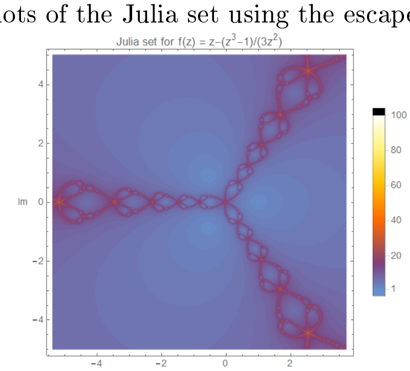


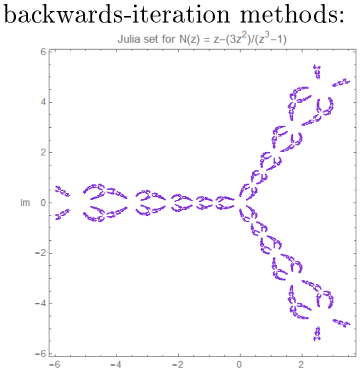




   - The Julia set appears to be $\boxed{\text{disconnected}}$, but not a Cantor set.
   - It appears that the filled Julia set has a number of separate connected region-like components, each of which seems to be separated by some distance from the others, and which is bounded by a curve.

   (c) $N(z) = z - \dfrac{z^3 - 1}{3z^2}$.

   - Here are plots of the Julia set using the escape-time and backwards-iteration methods:



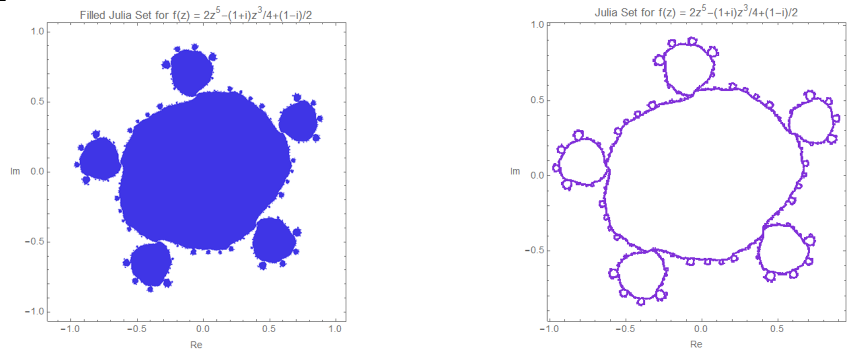




   - It is hard to tell, but based on the escape-time plot (which seems more conclusive), the Julia set appears to be $\boxed{\text{connected}}$.
   - Notice that this function is the Newton iterating function for $f(z) = z^3 - 1$, which has three roots in the complex plane given by 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$. This helps explain the obvious $2\pi/3$ rotational symmetry of the Julia set.

- The Julia sets for Newton iterating functions tend to be very interesting: each of the attracting basins of the fixed points has boundary equal to the Julia set, which produces very unusual pictures when the degree is larger than 2.

(d) $f(z) = 2z^5 - (1+i)z^3/4 + (1-i)/2$.

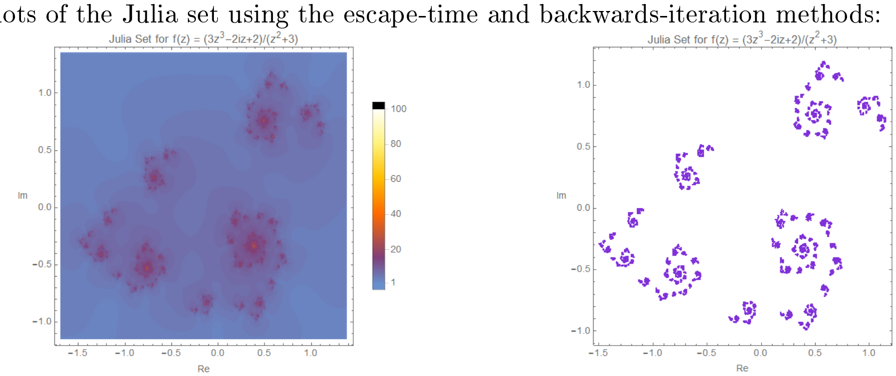
- Here are the plots of the filled Julia set and Julia set:




- Based on both plots the Julia set clearly seems $\boxed{\text{connected}}$.

(e) $f(z) = (3z^3 - 2iz + 2)/(z^2 + 3)$.

- Here are plots of the Julia set using the escape-time and backwards-iteration methods:




- The Julia set appears to be a $\boxed{\text{disconnected}}$ Cantor set.

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5. Plot their Julia sets for four holomorphic functions of your choice. In your response, please include the functions and also the Julia set plots.

- This is an open-ended problem. For some examples you can see the plots in problems 3 and 4.

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6. Consider the holomorphic function $f_k(z) = z^k$, where $k \geq 2$ is an integer.

(a) Show that, except for $z = 0$, every eventually periodic point of $f_k$ lies on the unit circle $|z| = 1$.

- If $f_k^a(z) = f_k^b(z)$ with $a < b$, then this implies $z^{k^a} = z^{k^b}$, which is in turn equivalent to $z^{k^a}(z^{k^b - k^a} - 1) = 0$.
- The solutions to this equation are $z = 0$ and the roots of $z^{k^b - k^a} = 1$, and by taking absolute values of both sides of the latter equation we immediately deduce $|z| = 1$.
- Thus, every eventually periodic point of $f_k$ lies on the unit circle.

(b) Show that every periodic cycle for $f_k$, except for the fixed point $z = 0$, is repelling.

- Suppose $\{z_1, z_2, \cdots, z_d\}$ is periodic, and not the fixed point $z = 0$. By part (a), each $|z_i| = 1$, so we have $|f_k'(z_i)| = \left|k\, z_i^{k-1}\right| = k$.
- Therefore, the product $\prod_{i=1}^{d} |f_k'(z_i)| = k^d > 1$, so the cycle is repelling.

(c) If $z = e^{2\pi it}$ where $t \in [0,1]$, show that $z$ is eventually periodic for $f_k$ if and only if $t$ is a rational number.

- The point is that the map $f_k$ is conjugate (by considering the value of $t$ modulo 1) to the multiplication-by-$k$ map on the interval $[0,1]$ modulo 1. We can then use the same arguments as those used all the way back on homework 1 (!).
- Explicitly, we have $f^a(z) = e^{2\pi it \cdot k^a}$ and $f^b(z) = e^{2\pi it \cdot k^b}$, and these are equal if and only if $2\pi it(k^b - k^a)$ is an integer multiple of $2\pi i$, which is to say that $t(k^b - k^a)$ is an integer, or that $t$ is rational.
- Conversely, if $t = \dfrac{p}{q}$ is rational, then $f_n(e^{2\pi i(p/q)}) = e^{2\pi i(a/q)}$ where $a = np \bmod q$. Thus, all the iterates of $t$ will lie in the finite set $\{1, e^{2\pi i(1/q)}, e^{2\pi i(2/q)}, \cdots, e^{2\pi i((q-1)/q)}\}$, and so eventually they must begin repeating, meaning that $t$ is eventually periodic.

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7. The goal of this problem is to explain some symmetries in the Julia sets we have examined.

(a) Show that the Julia set for any map in the quadratic family $q_c(z) = z^2 + c$ is symmetric about the origin.

- Since $q_c(z) = q_c(-z)$, the orbit of $-z$ is bounded if and only if the orbit of $z$ is bounded.
- Thus, the filled Julia set is symmetric about the origin. Since the Julia set is the boundary of the filled Julia set, it will also be symmetric about the origin.

(b) Show that the Julia set for the map $q_c(z) = z^2 + c$ is the reflection across the real axis of the Julia set for the map $q_{\overline{c}}(z) = z^2 + \overline{c}$.

- Observe that $\overline{q_c(z)} = \overline{z}^2 + \overline{c} = q_{\overline{c}}(\overline{z})$, so the orbit of $z$ under $q_c$ is bounded if and only if the orbit of $\overline{z}$ is bounded under $q_{\overline{c}}$.
- Thus, $z$ lies in the filled Julia set for $q_c$ if and only if $\overline{z}$ lies in the filled Julia set for $q_{\overline{c}}$, meaning that they are reflections across the real axis of one another.
- Then as in part (a), the same must hold for their boundaries, so the Julia sets are also reflections.

(c) Deduce that when $c$ is a real number, the Julia set for the map $q_c(z) = z^2 + c$ is symmetric about the real and imaginary axes.

- If $c$ is real then $q_{\overline{c}}(z) = q_c(z)$.
- So by part (a), the Julia set is symmetric about the origin, and by part (b) it is symmetric about the real axis. Combining these two statements shows it is symmetric about the imaginary axis as well.

(d) Show that the Julia set for the function $N(z) = z - \dfrac{z^3 - 1}{3z^2}$ from problem 4(c) has a $2\pi/3$-radian rotational symmetry around the origin. [Hint: For $\omega = e^{2\pi i/3}$, show that $N(\omega z) = \omega N(z)$.]

- Per the hint, letting $\omega = e^{2\pi i/3}$ so that $\omega^3 = 1$ we see $N(\omega z) = \omega z - \dfrac{\omega^3 z^3 - 1}{3\omega^2 z^2} = \omega z - \omega \dfrac{z^3 - 1}{z^2} = \omega N(z)$.
- Iterating, we have $N^n(\omega z) = \omega N^n(z)$. Therefore, if the orbit of $z$ is bounded, then so is the orbit of $\omega z$, because the points in its orbit are just $\omega$ times the points in the orbit of $z$.
- This means $z$ lies in the filled Julia set if and only if $\omega z$ does, and so as in the previous parts, the same holds for the boundary of the filled Julia set.
- But since multiplication by $\omega$ corresponds to a $2\pi/3$-radian counterclockwise rotation around the origin, that means the Julia set has a $2\pi/3$-radian rotational symmetry, as claimed.

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