E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 10, due Wed Apr 9th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the fixed points of each complex function and classify them as attracting, repelling, or neutral:

   (a) $f(z) = z + z^2(z - i)(z + 1)$.
   (b) $f(z) = z^2 - 3z + 5$.
   (c) $f(z) = z^2 - 2iz + (i - 1)$.
   (d) $f(z) = iz^2 + z + i/4$.
   (e) $f(z) = 2z^3 + 2z$.
   (f) $f(z) = 1 + \dfrac{4i}{z + 2 - 3i}$.

---

2. For each function $f(z)$, the given value z_0 is a periodic point. Find its period and classify the associated cycle as attracting, repelling, or neutral:

   (a) $f(z) = 1 - i + iz$ with $z_0 = 2$.
   (b) $f(z) = z^2 + i$ with $z_0 = -i$.
   (c) $f(z) = 1 - \dfrac{3}{2}z^2 - \dfrac{1}{2}z^3$ with $z_0 = 0$.
   (d) $f(z) = 3z + 4/z$ with $z_0 = i$.
   (e) $f(z) = z^2$ with $z_0 = \cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9} = e^{2\pi i/9}$.

---

3. For the following quadratic functions, (i) plot the Julia set / filled Julia set, (ii) use the picture to identify whether the Julia set for that function is connected or disconnected, and then (iii) justify your answer by computing the orbit of the critical point and using the fundamental dichotomy.

   (a) $f(z) = z^2 - 0.4 - 0.1i$.
   (b) $f(z) = z^2 + 0.2 - i$.
   (c) $f(z) = z^2 + i$.
   (d) $f(z) = z^2 - 0.53 + 0.6i$.

---

4. For the following non-quadratic functions, (i) plot the Julia set and filled Julia set and (ii) use the picture to identify whether the Julia set for that function seems to be connected or disconnected.

   (a) $f(z) = z^3 + 0.4z - i$.
   (b) $f(z) = \dfrac{z^3 - 1}{z + i}$.
   (c) $N(z) = z - \dfrac{z^3 - 1}{3z^2}$.
   (d) $f(z) = 2z^5 - (1 + i)z^3/4 + (1 - i)/2$.
   (e) $f(z) = (3z^3 - 2iz + 2)/(z^2 + 3)$.

---

5. Plot their Julia sets for four holomorphic functions of your choice. In your response, please include the functions and also the Julia set plots.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

6. Consider the holomorphic function $f_k(z) = z^k$, where $k \geq 2$ is an integer.

   (a) Show that, except for $z = 0$, every eventually periodic point of f_k lies on the unit circle $|z| = 1$.

   (b) Show that every periodic cycle for f_k, except for the fixed point $z = 0$, is repelling.

   (c) If $z = e^{2\pi i t}$ where $t \in [0, 1]$, show that z is eventually periodic for f_k if and only if t is a rational number.

---

7. The goal of this problem is to explain some symmetries in the Julia sets we have examined.

   (a) Show that the Julia set for any map in the quadratic family $q_c(z) = z^2 + c$ is symmetric about the origin.

   (b) Show that the Julia set for the map $q_c(z) = z^2 + c$ is the reflection across the real axis of the Julia set for the map $q_{\overline{c}}(z) = z^2 + \overline{c}$.

   (c) Deduce that when c is a real number, the Julia set for the map $q_c(z) = z^2 + c$ is symmetric about the real and imaginary axes.

   (d) Show that the Julia set for the function $N(z) = z - \dfrac{z^3 - 1}{3z^2}$ from problem 4(c) has a $2\pi/3$-radian rotational symmetry around the origin. [Hint: For $\omega = e^{2\pi i/3}$, show that $N(\omega z) = \omega N(z)$.]
