E. Dummit's Math 3543 ~ Dynamics, Chaos, and Fractals, Spring 2025 ~ Homework 1, due Wed Jan 15th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly and submit via Gradescope, making sure to select page submissions for each problem.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $g(x) = x^2$.
   (a) Compute the first four points on the orbit of 1, on the orbit of -2, and on the orbit of $1/2$.
   (b) Compute $g^2(x)$, $g^3(x)$, and $g^4(x)$. What is the general formula for $g^n(x)$?
   (c) Find all real numbers x that are eventually periodic points for g.

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2. For each function f and each point below, identify whether the point is (i) periodic, (ii) eventually periodic, or (iii) non-periodic.
   (a) $f(x) = x^2 - 2$, with points $x = \sqrt{2}$, $x = \sqrt{5}$, $x = -1$, $x = \dfrac{1+\sqrt{5}}{2}$.
   (b) $f(x) = \dfrac{1}{3}(3 + 5x - 2x^3)$, with points $x = \dfrac{1}{2}\sqrt{10}$, $x = -3$, $x = 0$, and $x = 3$.
   (c) $f(x) = |2x - 2| - x$, with points $x = 5$, $x = 10$, $x = \dfrac{5}{3}$, $x = -\dfrac{1}{5}$.

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3. Find all real fixed points (if there are any) for the following functions, and classify each of them as attracting, repelling, or neutral.
   (a) $a(x) = x^2 - 4x + 4$.
   (b) $b(x) = x^2 + 4$.
   (c) $c(x) = x^2 - \dfrac{1}{2}x + \dfrac{1}{2}$.
   (d) $d(x) = x^3$.
   (e) $e(x) = \dfrac{10}{x^2 + 1}$.
   (f) $f(x) = \dfrac{2x^3}{3x^2 - 1}$.
   (g) $g(x) = x \cos(x)$.
   (h) $h(x) = |x|$.

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4. For each of the following functions $f(x)$, the point $x = 0$ lies in a periodic orbit. Classify this orbit as attracting, repelling, or neutral:
   (a) $f(x) = 1 - \dfrac{3}{2}x^2 - \dfrac{1}{2}x^3$.
   (b) $f(x) = 5 - x$.
   (c) $f(x) = 2 + 2x - \dfrac{1}{4}x^2 - \dfrac{1}{4}x^3$.
   (d) $f(x) = \sqrt{2}(x^2 - x) + (1 - x)$.
   (e) $f(x) = 1 + 0.1x + 2.1x^2 - 1.2x^3$.
   (f) $f(x) = -\dfrac{4}{\pi}\tan^{-1}(x + 1)$.

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5. Find the two fixed points and the unique 2-cycle for the function $p(x) = x^2 - 7$, and classify each of them as attracting, repelling, or neutral.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

6. Let $m(x) = \dfrac{1}{1-x}$, defined for $x \neq 1$.
   (a) Does $m(x)$ have any real fixed points? What about $m^2(x)$?
   (b) Find the first ten elements in the orbit of $x = 2$ under m.
   (c) Show that every value of x (except $x = 0$ and $x = 1$) lies on a 3-cycle under m.

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7. Let $f(x) = e^x - \tan(x)$.
   (a) Show that $f(x)$ has a fixed point in the interval $(1.0, 1.1)$.
   (b) Show that $f(x)$ does not have a fixed point in the interval $(0.7, 0.8)$. [Hint: Show that $f(x) - x$ is positive on this interval.]
   (c) Show that $f(x)$ has a 2-cycle $\{\alpha, \beta\}$ where $0.7 < \alpha < 0.8$.

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8. Let $f : [0,1) \to [0,1)$ be defined as $f(x) = \begin{cases} 3x & \text{if } 0 \leq x < 1/3 \\ 3x-1 & \text{if } 1/3 \leq x < 2/3 \\ 3x-2 & \text{if } 2/3 \leq x < 1 \end{cases}$. Observe that $f(x) = 3x$ modulo 1, which is also equivalent to saying that $f(x) = \{3x\}$ is the fractional part of $3x$.
   (a) Sketch the graph of $y = f(x)$ and compare it to the doubling function $D(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x-1 & \text{if } 1/2 \leq x < 1 \end{cases}$.
   (b) Find all the fixed points of f.
   (c) Describe the orbits of $x = \frac{1}{3}, \frac{1}{4}, \frac{1}{7}$, and $\frac{1}{45}$ under f.
   (d) Suppose $x = \frac{p}{q}$ is a rational number. Show that x is either periodic or eventually periodic for f. [Hint: Consider the numerator and denominator of $f(x)$.]
   (e) Suppose x is a periodic point of exact period k. Show that x must be a rational number and in fact that the denominator of x divides $3^k - 1$. [Hint: Use the fact that $f^n(x) - 3^n x$ is an integer.]
   (f) Find five points that are periodic of exact order 3 for f.
