Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find all ordered pairs (a, b) of positive integers for which $\frac{1}{a} + \frac{2}{b} = \frac{1}{10}$.

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2. Find all ordered pairs (a, b) of positive integers for which $\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$.

   Remark: This is problem A1 from the 2018 Putnam exam.

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3. Find all solutions to the Diophantine equation $y^2 = x^4 + 2x^3 + 2x^2 + 4$.

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4. Find all integers n for which $n^3 - 10n^2 + 20n + 17$ is the cube of an integer.

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5. Let $n \geq 2$ be a fixed integer. Find infinitely many distinct positive integer triples (x, y, z) such that $x^n + y^n = z^{n+1}$. [Hint: Divide both sides by z^n.]

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

6. The goal of this problem is to solve the equation $x^y = y^x$ in positive rational numbers. Assume $x, y > 0$.
   (a) Prove that any rational solution with $y > x$ is of the form $(x, y) = ((1 + 1/u)^u, (1 + 1/u)^{u+1})$ for some rational number $u > 0$. [Hint: If $y > x$, set $y = (1 + 1/u)x$.]
   (b) Let $m \geq 2$. Show that the difference between any two positive consecutive mth powers is greater than m.
   (c) With notation as in part (a), suppose $u = n/m$ in lowest terms. Show that $m + n$ and n must both be mth powers and deduce that $m = 1$. [Hint: Write out x in terms of m, n and use the fact that $m+n, m, n$ are relatively prime.]
   (d) Conclude that the rational solutions to $x^y = y^x$ are of the form $(x, y) = (s, s)$ for rational s along with $(x, y) = ((1 + 1/n)^n, (1 + 1/n)^{n+1})$ or $((1 + 1/n)^{n+1}, (1 + 1/n)^n)$ for integers n.
   (e) Find all integral solutions to $x^y = y^x$.

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7. Prove that the sum of the first n positive integers is a perfect square for infinitely many values of n, and find the first five such n.

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8. Prove that there are no integral solutions to the equation $x^2 + y^2 = 3z^2$ other than $(0, 0, 0)$. [Hint: Use a descent argument modulo 3.]

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9. Prove that there are no integral solutions to the equation $y^9 = x^2 + 2024^{2020}$. [Hint: Work modulo 19.]

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10. The goal of this problem is to find all solutions to the Diophantine equation $x^2 + y^2 = 2z^2$ in various ways. If $\gcd(x, y, z) = d$ then clearly $(x/d, y/d, z/d)$ is also a solution, so it suffices to find all primitive solutions, in which $\gcd(x, y, z) = 1$. So now suppose that (x, y, z) is a primitive solution to $x^2 + y^2 = 2z^2$.

    (a) Show that x and y must have the same parity. Letting $a = (x - y)/2$ and $b = (x + y)/2$, show that there exist integers s, t such that a and b equal $2st$ and $s^2 - t^2$ in some order.

    (b) In the ring of Gaussian integers $\mathbb{Z}[i]$, show that $1 + i$ must divide both $x + iy$ and $x - iy$. Letting $p + iq = (x + iy)/(1 + i)$, show that there exist integers s, t such that p and q equal $s^2 - t^2$ and $2st$ in some order.

    (c) Show that the line through $(x/z, y/z)$ and $(-1, 1)$ has rational slope. Also, if ℓ is the line with rational slope t/s through the point $(-1, 1)$, find the intersection point of ℓ with the circle $(x/z)^2 + (y/z)^2 = 2$.

    (d) Find all primitive solutions to the Diophantine equation $x^2 + y^2 = 2z^2$.

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11. [Challenge] The goal of this problem is to give two ways to solve the Diophantine equation $(x^2 - xy - y^2)^2 = 1$ in positive integers. Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$, and let $\varphi = \dfrac{1 + \sqrt{5}}{2}$.

    (a) Show that $(x, y) = (F_{n+1}, F_n)$ satisfies $(x^2 - xy - y^2)^2 = 1$ for every $n \geq 1$.

    We now show that the pairs of consecutive Fibonacci numbers are the only solutions. The first approach is via a descent argument.

    (b) Suppose $(x, y) = (a, b)$ is a solution to $(x^2 - xy - y^2)^2 = 1$. Show that $a \geq b$, and that if $a > b$ then $(x, y) = (b, a - b)$ is also a solution to the system.

    (c) Prove that every solution to $(x^2 - xy - y^2)^2 = 1$ is of the form $(a, b) = (F_{n+1}, F_n)$ for some $n \geq 1$. [Hint: If $x > y$ use (b) to construct a smaller solution. Conclude the result via a descent/induction argument.]

    Now we give a second approach based on rational approximations.

    (d) Suppose that (x, y) is a solution to $\left|x^2 - xy - y^2\right| = 1$ with $x \geq y > 1$. Show that $\left|\dfrac{x}{y} - \varphi\right| < \dfrac{1}{2y^2}$. [Hint: Let $t = \dfrac{x}{y} - \dfrac{1}{2}$ and then show that $t > \frac{\sqrt{5}}{2} + \frac{1}{2y^2}$ and $t < \frac{\sqrt{5}}{2} - \frac{1}{2y^2}$ both yield contradictions.]

    (e) Deduce that every solution to the system is of the form $(x, y) = (F_{n+1}, F_n)$.
