E. Dummit's Math 4527 $\sim$ Number Theory 2, Spring 2024 $\sim$ Homework 2, due Fri Jan 26th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Express the following continued fractions as real numbers:
   (a) $[3, 1, 4, 1, 5]$.
   (b) $[\overline{1, 2, 3}]$.
   (c) $[\overline{3, 2, 1}]$.
   (d) $[\overline{3, 1, 2}]$.
   (e) $[3, \overline{1, 2}]$.
   (f) $[1, 2, \overline{1, 9, 1}]$.

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2. Find the continued fraction expansion, and the first five convergents, for each of the following:
   (a) $\sqrt{3}$.
   (b) $\sqrt{11}$.
   (c) $\dfrac{4+\sqrt{13}}{5}$.

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3. Find the rational number with denominator less than N closest to each of the following real numbers α:
   (a) $\alpha = \sqrt{13}$, $N = 100$.
   (b) $\alpha = \sqrt{2}$, $N = 100$.
   (c) $\alpha = e$, $N = 10000$. [Hint: See problem 8 for the continued fraction expansion of e.]

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. Let $\alpha = [\overline{3}] = [3, 3, 3, 3, 3, 3, 3, \dots]$.
   (a) Find α.
   (b) The first convergent to α is 3. Find the next five convergents to α.
   (c) Show that the nth convergent to α is the ratio s_n/s_{n-1} where $s_0 = 1$, $s_1 = 3$, and for $n \geq 2$, $s_{n+1} = 3s_n + s_{n-1}$.
   (d) For the sequence defined in (c), show that $\lim_{n\to\infty} s_{n+1}/s_n = \alpha$.

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5. The goal of this problem is to give a method for manipulating continued fractions using linear algebra. So suppose $a_0, a_1, \dots, a_n, \dots$ is a sequence of positive integers and set $p_n/q_n = [a_0, a_1, \dots, a_n]$ for each n.
   (a) Prove that $\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$.
   (b) Show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$. [Hint: Determinant.]
   (c) Show that $[a_n, a_{n-1}, \dots, a_0] = p_n/p_{n-1}$ and that $[a_n, a_{n-1}, \dots, a_1] = q_n/q_{n-1}$. [Hint: Transpose.]

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6. Let α be the real number $\alpha = \sum_{n=1}^{\infty} \frac{n^2}{100^n} = 0.0104091625364964....$

(a) Find the rational number p/q with smallest denominator that agrees with the expansion of α above, to the 16 decimal places shown.

(b) Prove that your answer in part (a) is the only rational number with denominator less than 10^{10} whose decimal expansion agrees with that of α to the 16 decimal places shown. [Hint: If $|p/q - a/b| < 10^{-16}$ then $|pb - aq| < 10^{-16} bq$.]

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7. The goal of this problem is to construct uncountably many transcendental numbers.

(a) Prove that the real number $\alpha = \sum_{k=1}^{\infty} \frac{1}{(k!)^{k!}} = 1 + \frac{1}{2^2} + \frac{1}{6^6} + \frac{1}{24^{24}} + \cdots$ is transcendental.

(b) Generalize (a) by showing that any number of the form $\beta = \sum_{k=1}^{\infty} \frac{(-1)^{s_k}}{(k!)^{k!}}$ is transcendental, for any choice of signs $(-1)^{s_k} = \pm 1$ on each term.

(c) Show that all of the numbers in (b) are distinct and lie in $(-2, 2)$. Deduce that there are uncountably many transcendental numbers in the interval $(-2, 2)$. [Hint: For distinctness, suppose that two of the numbers in (b) are equal and have their first unequal signs in the dth term. Show that the tails of the series are too small to account for the difference.]

(d) Conclude in fact that there are uncountably many transcendental numbers in any open interval (a, b) for any $a < b$.

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8. [Challenge] The goal of this problem is to obtain the continued fraction expansion $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \ldots]$. Let $\beta = [1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$ be the real number with continued fraction terms $a_{3i} = 1$, $a_{3i+1} = 2i$, and $a_{3i+2} = 1$ for each $i \geq 0$, and let $C_i = p_i/q_i$ be its convergents.

(a) Show that the convergents $C_i = p_i/q_i$ have numerators and denominators satisfying the recurrences

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2} & q_{3n} &= q_{3n-1} + q_{3n-2} \\ p_{3n+1} &= 2np_{3n} + p_{3n-1} & q_{3n+1} &= 2nq_{3n} + q_{3n-1} \\ p_{3n+2} &= p_{3n+1} + p_{3n} & q_{3n+2} &= q_{3n+1} + q_{3n} \end{aligned}$$

with initial values $p_0 = p_1 = q_0 = q_2 = 1$, $q_1 = 0$, and $p_2 = 2$.

(b) Now define the integrals $A_n = \int_0^1 \frac{x^n(x-1)^n}{n!} e^x \, dx$, $B_n = \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x \, dx$, $C_n = \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x \, dx$. Show that $A_n = -B_{n-1} - C_{n-1}$, $B_n = -2nA_n + C_{n-1}$, and $C_n = B_n - A_n$, and also that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = 0$. [Hint: For the first two, compute the derivatives of $\frac{1}{n!}x^n(x-1)^n e^x$ and $\frac{1}{n!}x^n(x-1)^{n+1}e^x$ and then integrate both sides from $x = 0$ to $x = 1$.]

(c) With notation as in part (b), show that $A_n = -(p_{3n} - q_{3n}e)$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$. [Hint: Show that A_n, B_n, C_n satisfy the same recurrences as the given combinations of p_n, q_n and also have the same initial values.]

(d) Conclude that $\beta = \lim_{i\to\infty} p_i/q_i = e$, and from this fact deduce the continued fraction expansion of e.

Remark: This argument was originally given by Hermite, and is adapted from an expository article of H.A. Cohn.
