E. Dummit's Math 4527 ~ Number Theory 2, Spring 2024 ~ Homework 12, due Fri Apr 12th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the number of Dirichlet characters (a) modulo 5, and (b) modulo 8, and compute their values explicitly.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

2. The Carmichael Λ-function $\Lambda(n)$ is defined to be $\ln(p)$ if $n = p^k$ is a prime power and 0 otherwise. It is frequently used in proofs of the prime number theorem.

   (a) Show that $\sum_{d|n} \Lambda(d) = \ln n$.

   (b) Show that the Dirichlet series for Λ is $D_\Lambda(s) = -\zeta'(s)/\zeta(s)$ for $\text{Re}(s) > 1$.

   (c) Show that $D_\Lambda(s) = \sum_{p \text{ prime}} \dfrac{\ln p}{p^s - 1}$ for $\text{Re}(s) > 1$. [Hint: Use (b) and logarithmic differentiation.]

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3. If S is a set of positive integers, its natural density is defined to be the value $\delta(S) = \lim_{N\to\infty} \dfrac{\#[S \cap \{1, 2, 3, \dots, N\}]}{N}$, if the limit exists.

   (a) Show that the natural density of the set of even integers is equal to 1/2.

   (b) Show that the natural density of the set of perfect squares is equal to 0.

   (c) Show that the natural density of any finite set of positive integers is equal to 0.

   (d) Show that the natural density of the set of integers with leading digit 1 (in base 10) is undefined. [Hint: Show that the ratio is at least 50% at $N = 2 \cdot 10^d$ and at most 20% at $N = 10^{d+1}$, for any d.]

   Remark: Note that this problem's version of natural density is not exactly the one we use in class, since the one used in class was only for sets of primes, relative to the set of all primes. (This problem is posed for all integers since it is easier to use this notion of natural density.)

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4. Let G be a finite abelian group with dual group $\hat{G}$, and recall the inner products $\langle f_1, f_2\rangle_G = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}$ on functions $f : G \to \mathbb{C}$ and $\langle \hat{f}_1, \hat{f}_2\rangle_{\hat{G}} = \frac{1}{|G|}\sum_{\chi\in\hat{G}} \hat{f}_1(\chi)\overline{\hat{f}_2(\chi)}$ on functions $\hat{f} : \hat{G} \to \mathbb{C}$. Also recall the Fourier transform of a function $f : G \to \mathbb{C}$ is the function $\hat{f} : \hat{G} \to \mathbb{C}$ with $\hat{f}(\chi) = \langle f, \chi\rangle_G = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{\chi(g)}$, and recall the Fourier inversion formula $f(g) = \sum_{\chi\in\hat{G}} \hat{f}(\chi)\chi(g)$ for each $g \in G$.

   (a) Prove Plancherel's theorem: $\frac{1}{|G|}\langle f_1, f_2\rangle_G = \langle \hat{f}_1, \hat{f}_2\rangle_{\hat{G}}$ for any functions $f_1, f_2 : G \to \mathbb{C}$. [Hint: Write $\hat{f}_1(\chi)$ as a sum over $g \in G$ and $\hat{f}_2(\chi)$ as a sum over $h \in G$, then use the fact that $\sum_{\chi\in\hat{G}} \overline{\chi(g)}\chi(h)$ is either $|G|$ or 0 according to whether $g = h$ or not.]

   (b) Deduce Parseval's theorem: $\frac{1}{|G|}\sum_{g\in G} |f(g)|^2 = \sum_{\chi\in\hat{G}} |\hat{f}(\chi)|^2$.

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5. Two primes that differ by 2 are called twin primes. It is conjectured that there are infinitely many twin primes, although interestingly, while the sum $\sum_{p \text{ prime}} \frac{1}{p}$ diverges, the sum $\sum_{p,p+2 \text{ prime}} \frac{1}{p} + \frac{1}{p+2}$ converges (the value of this series is known as Brun's constant, and is approximately 1.90216). Prove that there are infinitely many primes p that are *not* twin primes. [Hint: Apply Dirichlet.]

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6. The goal of this problem is to evaluate some Dirichlet L-series at 1.

   (a) Let χ_4 be the nontrivial Dirichlet character mod 4. Show $L(1,\chi_4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

   (b) Let $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x| < 1$. Show that $F'(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$ and deduce that $L(1,\chi_4) = F(1) = \int_0^1 \frac{1}{1+x^2}\,dx = \pi/4$. [Hint: Since the series for F converges absolutely, it can be differentiated term by term.]

   (c) Let χ_3 be the nontrivial Dirichlet character modulo 3. Show that $L(1,\chi_3) = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}$.

   (d) Let $G(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+1)(3n+2)}$ for $|x| < 1$. Show that $G(1) = \int_0^1 \int_0^y \frac{1}{1-x^3}\,dx\,dy$ and use this to compute the value of $L(1,\chi_3)$. [Hint: Note that $G''(x) = (1-x^3)^{-1}$ for $|x| < 1$. For the integral, change the order of integration.]

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7. [Challenge] Let p be a prime and let χ be the Legendre symbol modulo p. The goal of this problem is to evaluate $L(1,\chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k}$ explicitly, thus generalizing the calculations in problem 7, and then to use this evaluation to prove a formula for the class number in terms of the number of quadratic residues and nonresidues on the interval $[1,(p-1)/2]$ when $p \equiv 3 \pmod 4$. Recall the Gauss sum $g(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta^n$ where $\zeta = e^{2\pi i/p}$ is a primitive pth root of unity, and the general Gauss sum $g_k(\chi) = \sum_{n=1}^{p-1} \chi(n)\zeta^{kn}$.

   (a) Show that $-\log(1-\zeta^n) = \sum_{k=1}^{\infty} \frac{1}{k}\zeta^{nk}$. (Note that this series only converges conditionally.)

   (b) Let $S = \sum_{n=1}^{p-1} \chi(n) \cdot [-\log(1-\zeta^n)]$. Prove that $S = g(\chi)L(1,\chi)$. [Hint: Use (a), switch summation order, and use the Gauss sum identity $g_k(\chi) = \chi(k)^{-1}g(\chi)$.]

   (c) Define $P = \frac{\prod_{n\in NR}(1-\zeta^n)}{\prod_{n\in QR}(1-\zeta^n)}$ where NR is the set of quadratic nonresidues modulo p and QR is the set of quadratic residues modulo p. Show that $P = \exp(g(\chi)L(1,\chi))$.

   (d) Find the value of $L(1,\chi)$ for the Legendre symbol modulo 3. [Hint: The result of (c) is easier to calculate with, unless you like complex logarithms.]

   (e) Show that if $p \equiv 3 \pmod 4$, so that $\chi(-1) = -1$, then $S = -\frac{i\pi}{p}\sum_{n=1}^{p-1} \chi(n) \cdot n$ where S is as defined in (b). [Hint: In (b), interchange n with $-n$ and add the two sums together.]

   (f) Show that when $p \equiv 3 \pmod 4$ and $p > 3$ we have $h(-p) = -\frac{1}{p}\sum_{n=1}^{p-1} \chi(n) \cdot n$. [Hint: Use the Gauss sum evaluation $g(\chi) = i\sqrt{p}$ and the analytic class number formula.]

   (g) Show that when $p \equiv 3 \pmod 4$ and $p > 3$ we have $h(-p) = \frac{1}{2-\chi(2)}\sum_{n=1}^{(p-1)/2} \chi(n)$. [Hint: Decompose $\sum_{n=1}^{p-1} \chi(n) \cdot n$ into two ranges in two different ways: one into even and odd, and another into $[1,(p-1)/2]$ and $p - [1,(p-1)/2]$.]

   (h) Deduce that when $p \equiv 3 \pmod 4$, the class number of $\mathcal{O}_{\sqrt{-p}}$ is equal to $\frac{1}{2-\chi(2)}$ times the number of quadratic residues in $[1,(p-1)/2]$ minus the number of quadratic nonresidues on that interval, so in particular there are always more quadratic residues than quadratic nonresidues. Also deduce in particular that this class number is always odd.

   (i) Find the class numbers of $\mathcal{O}_{\sqrt{-7}}$, $\mathcal{O}_{\sqrt{-11}}$, $\mathcal{O}_{\sqrt{-19}}$, and $\mathcal{O}_{\sqrt{-31}}$. [If you're still here at this point, for convenience $\chi(2) = 1$ when $p \equiv 7 \pmod 8$ and $\chi(2) = -1$ when $p \equiv 3 \pmod 8$.]
