E. Dummit's Math 4527 ~ Number Theory 2, Spring 2024 ~ Homework 11, due Fri Apr 5th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\Delta = -39$.

   (a) Calculate the Minkowski bound for $\mathcal{O}_{\sqrt{\Delta}}$ and determine the splitting of each prime ideal up to the bound.

   (b) Determine the structure of the ideal class group of $\mathcal{O}_{\sqrt{\Delta}}$ and identify an ideal representing each ideal class.

   (c) Compute the reduced positive-definite quadratic form corresponding to each of the ideal class representatives you selected in (b).

   (d) Find all of the reduced positive-definite quadratic forms of discriminant Δ.

   (e) Determine the structure of the ideal class group by computing Dirichlet compositions. (You may assume the norm form is the identity and that the inverse of $ax^2 + bxy + cy^2$ is represented by $ax^2 - bxy + cy^2$.)

   (f) Compute the ideal class of $\mathcal{O}_{\sqrt{\Delta}}$ corresponding to each of the reduced positive-definite forms.

   (g) Which method (ideals or forms) do you prefer? Why?

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2. Let N_k be the function with $N_k(n) = n^k$ for a positive integer k.

   (a) Find the Dirichlet series $D_{N_k}(s)$ in terms of the Riemann zeta function.

   (b) If σ_k is the sum-of-kth-powers-of-divisors function $\sigma_k(n) = \sum_{d|n} d^k$, find the Dirichlet series $D_{\sigma_k}(s)$ in terms of the Riemann zeta function.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Prove that a prime p is represented by the quadratic form $x^2 + xy + 2y^2$ if and only if $p = 7$ or $p \equiv 1, 2, 4 \pmod 7$.

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4. The goal of this problem is to prove a theorem of Euler about representations of integers by binary quadratic forms associated to $\mathcal{O}_{\sqrt{-5}}$.

   (a) If $p \neq 2, 5$ is a prime, show that p is represented by a binary quadratic form of discriminant $\Delta = -20$ if and only if $p \equiv 1, 3, 7, 9 \pmod{20}$.

   (b) Show that $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ are the only reduced positive-definite quadratic forms of discriminant $\Delta = -20$.

   (c) Show that $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ are not equivalent forms, and deduce that the class number of $\mathcal{O}_{\sqrt{-5}}$ is 2. [Hint: One form represents 2 while the other doesn't.]

   (d) Sharpen part (a) by showing that a prime $p \neq 2, 5$ is represented by $x^2 + 5y^2$ if and only if $p \equiv 1, 9 \pmod{20}$ and it is represented by $2x^2 + 2xy + 3y^2$ if and only if $p \equiv 3, 7 \pmod{20}$. [Hint: Show that p or $2p$ is a quadratic residue modulo 5, respectively.]

   (e) Prove that an integer n can be represented in the form $x^2 + 5y^2$ or $2x^2 + 2xy + 3y^2$ if and only if each prime dividing n to an odd power is equal to 2 or 5 or is congruent to $1, 3, 7, 9$ modulo 20. Show furthermore that it is of the form $x^2 + 5y^2$ if and only if the total number of primes dividing n to an odd power that are congruent to 2, 3, or 7 modulo 20 is even.

   (f) Prove that if an integer n can be written in the form $x^2 + 5y^2$ for rational numbers x, y then it can be written in that form for integers x, y.

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5. The goal of this problem is to provide another approach to showing that $\lim_{s\to 1+}(s-1)\zeta(s) = 1$ for s real.

   (a) Show that $\dfrac{1}{(n+1)^s} < \int_n^{n+1} x^{-s}\,dx < \dfrac{1}{n^s}$ for any positive integer n and any real $s > 1$.

   (b) Show that $\dfrac{1}{s-1} < \zeta(s) < 1 + \dfrac{1}{s-1}$ for any real $s > 1$.

   (c) Deduce that $\lim_{s\to 1+}(s-1)\zeta(s) = 1$.

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6. [Challenge] It was observed by Euler that certain quadratic polynomials of the form $f_n(t) = t^2 - t + n$ take only prime values for unexpectedly long ranges of values of t. For example, for $n = 11$ we can see that with $f_{11}(x) = t^2 - t + 11$ then the values of f_{11} for $t = 1, 2, \ldots, 10$ are respectively 11, 13, 17, 23, 31, 41, 53, 67, 83, and 101, all of which are prime.

   (a) For the polynomial $f_{41}(t) = t^2 - t + 41$, find the smallest positive integer t for which $f_{41}(t)$ is composite.

   Now suppose $p = 4n-1$ is a prime where $p \geq 7$. The goal now is to characterize these prime-valued polynomials in terms of the class number of $\mathcal{O}_{\sqrt{-p}}$. Observe that $-p$ is congruent to 1 modulo 4, so the only ramified prime in $\mathcal{O}_{\sqrt{-p}}$ is p.

   (b) Suppose the prime $q < n$ splits in $\mathcal{O}_{\sqrt{-p}}$ as $(q) = (q, \frac{1+\sqrt{-p}}{2} - r)(q, \frac{1+\sqrt{-p}}{2} - r')$ where $0 \leq r, r' \leq q-1$. Show that $r^2 - r + n$ is divisible by q and cannot equal q, and conclude that $r^2 - r + n$ is composite.

   (c) Conversely, suppose that $r^2 - r + n$ is composite for some $0 \leq r \leq n-2$. If q is the smallest prime dividing $r^2 - r + n$, show that $q < n$ and that the ideal (q) splits in $\mathcal{O}_{\sqrt{-p}}$.

   (d) Deduce that all of the integers $f_n(t) = t^2 - t + n$ with $0 \leq t \leq n-2$ are prime if and only if every prime $q < n$ is inert in $\mathcal{O}_{\sqrt{-p}}$.

   (e) If every prime $q < n$ is inert in $\mathcal{O}_{\sqrt{-p}}$, show that the class number $h(-p) = 1$. [Hint: Minkowski's bound.]

   (f) Conversely, if the class number $h(-p) = 1$, show that every prime $q < n$ is inert in $\mathcal{O}_{\sqrt{-p}}$. [Hint: Show that (q) cannot split by considering the norm of a factor, which is by hypothesis principal.]

   (g) Conclude that $f_n(t) = t^2 - t + n$ is prime for all integers $0 \leq t \leq n-2$ if and only if the class number $h(-p) = 1$.
