E. Dummit's Math 4527 ~ Number Theory 2, Spring 2024 ~ Homework 1, due Fri Jan 19th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Identify all pages containing each problem when submitting the assignment.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find all solutions in integers (if any) to the following linear Diophantine equations:
   (a) $22a + 17b = 19$.
   (b) $42a + 27b = 39$.
   (c) $3a + 7b + 16c = 8$.

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2. Find all right triangles having one leg of length 40, and whose other two side lengths are integers.

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3. Byzantine Basketball is like regular basketball except that foul shots are worth a points instead of two points and field shots are worth b points instead of three points. Moreover, in Byzantine Basketball there are exactly 35 scores that never occur in a game, one of which is 58. What are a and b?

   Remark: This problem was on the 1971 Putnam exam.

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4. Compute the following things:
   (a) Show that 7/13 and 13/24 are adjacent in the Farey sequence of level 24. What are the next three terms after them?
   (b) Find all n such that exactly 2 terms appear between 7/13 and 13/24 in the Farey sequence of level n.
   (c) List all the terms between 6/19 and 5/14 in the Farey sequence of level 19.
   (d) Find the three terms following 154/227 in the Farey sequence of level 2024.
   (e) List all the terms between 1502/1801 and 1492/1789 in the Farey sequence of level 2024.

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5. Find the continued fraction expansions for the following rational numbers:
   (a) 355/113.
   (b) 418/2021.
   (c) 13579/2468.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

6. Prove that the area of any right triangle with integer side lengths is always divisible by 6, but not necessarily any integer greater than 6.

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7. The goal of this problem is to discuss the Weierstrass substitution $t = \tan(\theta/2)$. Recall the trigonometric identities $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ shown in class. Let $t = \tan(\theta/2)$.

   (a) Show that $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, and $d\theta = \frac{2dt}{1+t^2}$.

   (b) Use the Weierstrass substitution to compute the indefinite integral $\int \frac{1}{5+3\cos\theta}\,d\theta$.

   (c) Use the Weierstrass substitution to compute the definite integral $\int_{-\pi/2}^{\pi/2} \frac{1}{4-\sin\theta}\,d\theta$.

   Remark: As can be seen from (b) and (c), the Weierstrass substitution allows evaluation of any integral that is a rational function of $\sin\theta$ and $\cos\theta$.

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8. Find the smallest positive integer n such that for all integers m with $0 < m < 2023$, there exists an integer k with $\frac{m}{2023} < \frac{k}{n} < \frac{m+1}{2024}$. (Make sure to prove that your value is the smallest possible.)

   Remark: This is a variation of problem B1 from the 1993 Putnam exam.

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9. For each Farey fraction a/b, define $\mathcal{C}(a/b)$ to be the circle in the upper-half of the Cartesian plane of radius $r_{a/b} = 1/(2b^2)$ that is tangent to the x-axis at the point $(a/b, 0)$. These circles are called the <u>Ford circles</u>.

   (a) If a/b and c/d are adjacent entries in some Farey sequence, prove that the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are tangent.

   (b) If a/b and c/d are not adjacent in any Farey sequence, prove that the interiors of the circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are disjoint.

   (c) Draw (you should probably use a computer) the 11 Ford circles corresponding to the Farey sequence of level 5.

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10. [Challenge] Let a, b, c be pairwise relatively prime positive integers. Show that $2abc - ab - bc - ca$ is the largest integer that cannot be expressed in the form $xbc + yca + zab$ for nonnegative integers x, y, z. [Hint: Any integer is congruent modulo a to precisely one of 0, bc, $2bc$, ... , $(a-1)bc$.]

    Remark: This is a Frobenius coin problem for the integers ab, bc, ca and was problem 3 from the 1983 IMO.
