
Directions: Read ALL of the following directions.

This is an open-notes, open-homework, open-textbook exam. There is no official time limit, but it is suggested that you should be able to solve most of the problems within approximately 15 hours.

There are 4 problems totaling 200 points on this exam.

Justify any answers you give, including computations. You may freely refer to results from in class, from the course notes, and the course assignments and solutions, but please make it clear what results you are using.

Proofs and explanations are expected to be clear, concise, and correct.

You MAY use a computer for typesetting and to access any material on the course webpage (e.g., the course notes and homework solutions), and to perform computations. Any such computations must be clearly identified and justified as correct.

You MAY NOT use a computer to access any other information.

You MAY ask the instructor for help on any part of the exam, during office hours or via email. (The instructor may or may not decide to grant help, but you are encouraged to ask regardless.)

You MAY NOT discuss the material on this exam with anyone except for the instructor (until after the due date). This includes asking others for hints or solutions, searching for information about the problems online, or posting about the problems on discussion forums.

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1. [20 points] Summarize the main topics covered in each of the four chapters of this course, and identify at least one interesting thing you learned from each of them.

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2. [20 points] What topic(s) in this course did you most enjoy? Which topic(s) did you least enjoy? Do you have any feedback about the course, the presentation, the organization of the material, or anything else?

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3. [$20 \times 3 = 60$ points] Solve up to twenty (20) of the following problems (if you do more than 20, the best 20 will be counted). Provide 1-2 sentences' worth of justification for how you did any relevant calculations.

(a) Find all right triangles having one leg of length 35 and whose other two side lengths are integers.

(b) Find the three terms following $1249/1780$ in the Farey sequence of level 2024.

(c) Find all the terms between $1480/1779$ and $1589/1910$ in the Farey sequence of level 2024.

(d) Find the real number values of $[1, \overline{3, 2}]$ and $[\overline{1, 5, 3}]$.

(e) Find the continued fraction expansions of $2 + \sqrt{3}$ and of $5 - \sqrt{6}$.

(f) Determine the rational number whose decimal expansion begins $0.53415453527435610302\ldots$ given that the denominator is less than 10^6.

(g) Briefly explain why $355/113 \approx 3.14159292$ is such a good approximation to π.

(h) Find the fundamental units of $\mathbb{Z}[\sqrt{65}]$ and $\mathbb{Z}[\sqrt{67}]$.

(i) Determine if $x^2 - 91y^2 = -1$ has integer solutions and find the smallest two solutions to the Pell equation $x^2 - 91y^2 = 1$.

(j) Use the super magic box to compute the fundamental unit of $\mathbb{Z}[\sqrt{1221}]$ and to factor 1221.

(k) Find the greatest common divisor of $17 + 9i$ and $9 + 15i$ in $\mathbb{Z}[i]$ and of $7 - 17\sqrt{-2}$ and $14 - 15\sqrt{-2}$ in $\mathbb{Z}[\sqrt{-2}]$.

(l) Find the prime factorizations of $17 + 9i$ in $\mathbb{Z}[i]$ and $9 - 11\sqrt{-3}$ in $\mathcal{O}_{\sqrt{-3}}$.

(m) Find the smallest squarefree integer > 1 that can be written in the form $a^2 + b^2$, and in the form $c^2 + 2d^2$, and in the form $e^2 + 3f^2$.

(n) Find all solutions to the Diophantine equation $x^2 + y^2 = z^4$ where x and y are relatively prime.

(o) Find the prime ideal factorizations of (2), (3), (5), and (7) in $\mathcal{O}_{\sqrt{-13}}$.

(p) Compute the cardinality of the quotient ring $\mathbb{Z}[\sqrt{11}]/(5, 1 + \sqrt{11})$ and find representatives for the residue classes in the quotient ring.

(q) Calculate the cubic residue symbol $\left[\dfrac{8 + \sqrt{-3}}{2 + 4\sqrt{-3}}\right]_3$ and the quartic residue symbol $\left[\dfrac{4 - i}{5 + 2i}\right]_4$.

(r) Calculate the splitting of each prime ideal of norm less than the Minkowski bound for $\mathcal{O}_{\sqrt{-67}}$.

(s) Find the class number of $\mathcal{O}_{\sqrt{-6}}$.

(t) Find the class number of $\mathcal{O}_{\sqrt{11}}$.

(u) Find reduced quadratic forms (properly) equivalent to $47x^2 - 83xy + 67y^2$ and to $27x^2 - 95xy + 84y^2$.

(v) Find all of the reduced positive-definite quadratic forms of discriminant $\Delta = -55$.

(w) Find the Dirichlet composition of the quadratic forms $2x^2 + 7y^2$ and $3x^2 - 2xy + 5y^2$ of discriminant $\Delta = -56$.

(x) Let f be the function with $f(n) = \sum_{d|n} \sqrt{d}$. Find the Dirichlet series for f in terms of the Riemann ζ-function.

(y) Find the number of Dirichlet characters modulo 7, and give their values explicitly.

(z) Verify the analytic class number formula for $D = -1$ and for $D = -3$.

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4. [$20 \times 5 = 100$ points] Solve up to twenty (20) of the following problems (if you solve more than 20, the best 20 will be counted). Justify all responses with rigorous proof.

(a) Prove that a right triangle with integer side lengths always has at least one side length divisible by 5.

(b) For a triangle of side lengths a, b, and c having area K, its inradius is equal to $\frac{2K}{a+b+c}$. Prove that a Pythagorean right triangle always has an integer inradius.

(c) Prove that there are infinitely many triangles whose side lengths are consecutive integers and whose area is also an integer, and give three examples of such triangles. [Hint: Heron's formula.]

(d) Prove that the real number $\alpha = \sum_{n=0}^{\infty} \frac{1}{2^{10^n}} = \frac{1}{2} + \frac{1}{2^{10}} + \frac{1}{2^{100}} + \frac{1}{2^{1000}} + \cdots$ is transcendental.

(e) Prove that there are no solutions to the Diophantine equation $x^5 - y^2 = 4$.

(f) Find all solutions to the Diophantine equation $x^2 - 2y^2 = z^2$.

(g) Prove that there are no integral solutions to the equation $x^2 + y^2 = 7z^2$ other than $(0, 0, 0)$.

(h) Suppose R is a finite commutative ring with 1. Prove that every prime ideal of R is maximal.

(i) Prove that the units in the quadratic integer ring $\mathbb{Z}[\sqrt{14}]$ are the elements $\pm(15 + 4\sqrt{14})^n$ for $n \in \mathbb{Z}$.

(j) Suppose I and J are ideals of a quadratic integer ring $\mathcal{O}_{\sqrt{D}}$. Prove that $IJ = (I + J)(I \cap J)$. [Hint: See problem 6 of homework 7.]

(k) Prove that 26 has two inequivalent irreducible factorizations in $\mathbb{Z}[\sqrt{-17}]$ that become equivalent when factored into ideals.

(l) In the quadratic integer ring $\mathcal{O}_{\sqrt{D}}$, if $r \in \mathcal{O}_{\sqrt{D}}$ has $N(r) = \pm p$ for a prime p, show that r is a prime element of $\mathcal{O}_{\sqrt{D}}$.

(m) Prove that the integer solutions to the equation $y^2 = x^3 - 4$ are $(x, y) = (2, \pm 2)$ and $(5, \pm 11)$.

(n) For any $\alpha, \beta \in \mathbb{R}$ and positive integer N, show that there exist integers p, q, r with $1 \le r \le N$ such that $|\alpha - p/r|$ and $|\beta - q/r|$ are both less than $\frac{1}{r\sqrt{N}}$. [Hint: Apply Minkowski's theorem to the set in $\mathbb{R}^3$ with $|x| \le N$, $|\alpha x - y| \le 1/\sqrt{N}$, $|\beta x - z| \le 1/\sqrt{N}$.]

(o) Prove that every positive integer is the sum of three triangular numbers.

(p) Prove that $\mathbb{Z}[\sqrt{7}]$ is a principal ideal domain.

(q) Prove that $\mathbb{Z}[\sqrt{10}]$ has class number 2.

(r) Prove that the ideal class group of $\mathcal{O}_{\sqrt{-59}}$ has order 3.

(s) Prove that $SL_2(\mathbb{Z})$ is generated by the matrices $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(t) Find all of the reduced positive-definite binary quadratic forms of discriminant -47 and identify the class group using Dirichlet composition.

(u) Find all of the reduced positive-definite binary quadratic forms of discriminant -14 and find an ideal in the ideal class corresponding to each one.

(v) Prove that a prime p is represented by the quadratic form $x^2 + xy + 3y^2$ if and only if $p = 11$ or $p \equiv 1, 3, 4, 5, 9 \pmod{11}$.

(w) Let p be a prime congruent to 1 modulo 8 and let $f(x, y) = 2x^2 + 2xy + \frac{p+1}{2}y^2$. Compute the Dirichlet composition of f with itself and use the result to show that the class group of $\mathcal{O}_{\sqrt{-p}}$ has even order.

(x) Let χ_5 be the mod-5 Dirichlet character with $\chi_5(2) = i$. Prove that the value of the associated L-series at $s = 1$ is $L(1, \chi_5) = \sum_{n=0}^{\infty} \frac{3}{(5n+1)(5n+4)} + i \sum_{n=0}^{\infty} \frac{1}{(5n+2)(5n+3)}$.

(y) Use the analytic class number formula to compute $L(1, \chi)$ where χ is the Jacobi symbol modulo 5. [Hint: Use $D = 5$.]

(z) Given that if p is a prime with $p \equiv 3 \pmod{4}$, the class number of $\mathcal{O}_{\sqrt{-p}}$ is equal to $\frac{1}{2 - \chi(2)}$ times the number of quadratic residues in $[1, (p-1)/2]$ minus the number of quadratic nonresidues on that interval, where χ is the Jacobi symbol mod p, prove that this class number is always odd.
