E. Dummit's Math 3527 $\sim$ Number Theory I, Spring 2024 $\sim$ Homework 2, due Tue Jan 23rd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Find the following:
   (a) The gcd and lcm of 288 and 600.
   (b) The prime factorizations of 2023 and 2024. (You may want to ask a computer to find these!)
   (c) The prime factorizations of 2023^{2024} and 2024^{2023}.
   (d) The prime factorizations of 111, 1001, and 111111.
   (e) The gcd and lcm of $2^8 3^{11} 5^7 7^8 11^2$ and $2^4 3^8 5^7 7^7 11^{11}$.
   (f) The prime factorization of 15! and the number of positive divisors of 15!.
   (g) The number of divisors and the sum of divisors of 10000.

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2. Find examples of the following things:
   (a) Four different pairs of positive integers (a, b) with $a \le b$ such that $\gcd(a, b) = 30$ and $\text{lcm}(a, b) = 1800$.
   (b) A positive integer n such that $n/2$ is a perfect square and $n/3$ is a perfect cube.
   (c) A positive integer n such that n, $n + 1$, $n + 2$, $n + 3$, $n + 4$, and $n + 5$ all have more than one distinct prime factor.
   (d) An integer that is a multiple of 15 that has exactly 15 positive divisors.

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3. It is sometimes claimed (occasionally in actual textbooks) that if $p_1, p_2, \dots, p_k$ are the first k primes, then the number $n = p_1 p_2 \cdots p_k + 1$ used in Euclid's proof is always prime for any $k \ge 1$. Find a counterexample to this statement; make sure to justify that it is actually a counterexample.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let n be a positive integer greater than 1.
   (a) Show that if n is composite, then n must have at least one divisor d with $1 < d \le \sqrt{n}$. Deduce that if n is composite, then n has at least one prime divisor $p \le \sqrt{n}$. [Hint: Write $n = ab$ where $1 < a \le b < n$.]
   (b) Show that if no prime less than or equal to $\sqrt{n}$ divides n, then n is prime.
   (c) Show explicitly that $n = 109$ and $n = 251$ are prime by verifying that they are not divisible by any prime $\le \sqrt{n}$.

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5. Recall that if N is a positive integer, then $\sigma(N)$ denotes the sum of the positive divisors of N. We say that N is a perfect number when $\sigma(N) = 2N$: this is often phrased as "the sum of all of the proper divisors of N equals N itself.

   (a) Show that if $2^n - 1$ is a prime number, then the number $N = 2^{n-1}(2^n - 1)$ is perfect.

   (b) Show that 28, 496, 8128 are perfect numbers.

   Remark: Perfect numbers have been of mathematical (and numerological) interest since antiquity. Euclid established the result in part (a), and roughly two millennia later, Euler showed that every even perfect number must be of the form described in (a). It is not known whether there are infinitely many even perfect numbers, and it is also not known whether there are any odd perfect numbers.

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6. The goal of this problem is to study which numbers of the form $N = a^k - 1$ can be prime, where a and k are positive integers greater than 1.

   (a) Show that $n^k - 1$ is divisible by $n - 1$, for any integer n.

   (b) Show that if $a > 2$, then $N = a^k - 1$ is not prime.

   (c) Show that if k is composite, then $N = 2^k - 1$ is not prime. [Hint: If $k = rs$, show N is divisible by $2^r - 1$.]

   (d) Deduce that the only possible primes of the form $N = a^k - 1$ are those of the form $2^p - 1$ where p is a prime. (Such primes are called Mersenne primes.) Are all the numbers of the form $2^p - 1$ (p prime) actually prime?

   Remark: Notice that Mersenne primes can be used to construct perfect numbers, as described in problem 4.

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7. Prove that $\log_3 7$ is irrational. [Hint: Suppose otherwise, so that $\log_3 7 = a/b$. Convert this to statement about positive integers and find a contradiction.]
