E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2023 ~ Homework 9, due Fri Mar 31st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a finite-dimensional vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) V has a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T if and only if T is diagonalizable.

   (b) If $\dim(V) = n$ and T has n distinct eigenvalues in F, then T is diagonalizable.

   (c) If $\dim(V) = n$ and T is diagonalizable, then T has n distinct eigenvalues in F.

   (d) If A is a diagonalizable $n \times n$ matrix, then so is $A + I_n$.

   (e) For any scalar λ, the λ-eigenspace of T is a subspace of the generalized λ-eigenspace of T.

   (f) For any λ, a chain of generalized λ-eigenvectors is linearly independent.

   (g) There always exists a basis β of V consisting of generalized eigenvectors of T.

   (h) If all eigenvalues of T lie in F, then there exists a basis β of V of generalized eigenvectors for T.

   (i) There always exists some basis β of V such that the matrix $[T]_\beta^\beta$ is in Jordan canonical form.

   (j) Every matrix $A \in M_{n\times n}(\mathbb{C})$ has a Jordan canonical form.

   (k) If a matrix is diagonalizable, then its Jordan canonical form is diagonal.

   (l) If the Jordan canonical form of a matrix is diagonal, then the matrix is diagonalizable.

   (m) Two matrices are similar if and only if they have equivalent Jordan canonical forms.

   (n) If J is the Jordan canonical form of A, then $J + I_n$ is the Jordan canonical form of $A + I_n$.

   (o) If J is the Jordan canonical form of A, then J^2 is the Jordan canonical form of A^2.

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2. Suppose the characteristic polynomial of the 5×5 matrix A is $p(t) = t^3(t-1)^2$.

   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.

   (b) Find the determinant and trace of A.

   (c) List all possible Jordan canonical forms of A up to equivalence.

   (d) If $\ker(A)$ and $\ker(A - I)$ are both 2-dimensional, what is the Jordan canonical form of A?

   (e) If A^3 is diagonalizable but A^2 is not, what is the Jordan canonical form of A?

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3. Find the Jordan canonical form of each matrix A over $\mathbb{C}$.

   (a) $A = \begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

   (e) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix}$.

   (f) $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix}$.

   (g) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. The goal of this problem is to find the eigenvalues and eigenvectors of the $n \times n$ "all 1s" matrix over an arbitrary field F. So let $n \geq 2$ and let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

   (a) Show that the 0-eigenspace of A has dimension $n-1$ and find a basis for it.

   (b) If the characteristic of F does not divide n, find the remaining nonzero eigenvalue of A and a basis for the corresponding eigenspace, and show that A is diagonalizable. [Hint: Calculate the trace of A.]

   (c) If the characteristic of F does divide n, show that A is not diagonalizable, and find its Jordan canonical form. [Hint: Note that $\text{char}(F)$ dividing n is the same as saying that $n = 0$ in F.]

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5. Suppose V is finite-dimensional and $T : V \to V$ is a projection, so that $T^2 = T$.

   (a) Show that the only possible eigenvalues of T are 0 and 1.

   (b) Show that T is diagonalizable. [Hint: See problem 5 of homework 6.]

   (c) Suppose A and B are projection maps on V of the same rank. Show that A and B are similar. Deduce that projection maps on V are characterized up to similarity by their rank.

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6. Suppose A is an invertible $n \times n$ matrix and that $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is its characteristic polynomial. Note that $a_0 = (-1)^n \det(A)$ is nonzero.

   (a) If $B = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n)$, show that $AB = I_n$. [Hint: Cayley-Hamilton.]

   (b) Show that there exists a polynomial $q(x)$ of degree at most $n-1$ such that $A^{-1} = q(A)$.

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7. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) Show that any Jordan-block matrix is similar to its transpose. [Hint: Reverse the Jordan basis.]

   (b) If J is a matrix in Jordan canonical form, show that J is similar to its transpose.

   (c) Show that A is similar to its transpose.

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8. [Challenge] The goal of this problem is to prove various results about eigenvalues of complex matrices and stochastic matrices. Let $A \in M_{n\times n}(\mathbb{C})$, define $\rho_i(A)$ to be the sum of the absolute values of the entries in the ith row of A, and define $\rho(A) = \max_{1\leq i\leq n} \rho_i(A)$.

   (a) Define the ith Gershgorin disk C_i to be the disc in $\mathbb{C}$ centered at $a_{i,i}$ with radius $r_i(A) = \rho_i(A) - |a_{i,i}|$. Prove Gershgorin's disc theorem: every eigenvalue of A is contained in one of the Gershgorin disks of A. [Hint: If $\mathbf{v} = (x_1, \ldots, x_n)$ is an eigenvalue where x_k has the largest absolute value among the entries of $\mathbf{v}$, show that $|\lambda x_k - a_{k,k}x_k| \leq r_i(A)\,|x_k|$ by noting that λx_k is the kth component of $A\mathbf{v}$.]

   (b) For any eigenvalue λ of $A \in M_{n\times n}(\mathbb{C})$, prove that $|\lambda| \leq \rho(A)$.

   (c) Prove that if $A \in M_{n\times n}(\mathbb{R})$ has positive entries and there exists an eigenvalue λ such that $|\lambda| = \rho(A)$, then $\lambda = \rho(A)$ and the λ-eigenspace is 1-dimensional and spanned by the vector $\mathbf{v} = (1, 1, \ldots, 1)$. [Hint: Analyze when equality can hold in (a) and (b).]

   (d) If M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1), show that every eigenvalue λ of M has $|\lambda| \leq 1$. Also show that if M has all entries positive, then the only eigenvalue of M of absolute value 1 is $\lambda = 1$, and the 1-eigenspace has dimension 1. [Hint: Consider M^T.]
