E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2023 ~ Homework 7, due Fri Mar 17th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V with scalar field F with $\mathbf{v}, \mathbf{w} \in V$, and let W be a subspace of V. Identify each of the following statements as true or false:

   (a) If V is a complex vector space, the vectors $\mathbf{v}$ and $i\mathbf{v}$ are always orthogonal.

   (b) $\frac{1}{9}(4, -1, 8)$, $\frac{1}{9}(7, -4, -4)$, $\frac{1}{9}(4, 8, 1)$ is an orthonormal basis of $\mathbb{R}^3$, with the standard dot product.

   (c) An orthogonal set of vectors is linearly independent.

   (d) An orthonormal set of vectors is linearly independent.

   (e) Every finite-dimensional inner product space has an orthonormal basis.

   (f) If V is finite-dimensional and W is any subspace of V, then $\dim(W) = \dim(W^\perp)$.

   (g) If V has an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, then $||\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4|| = 2$.

   (h) If $\mathbf{w}^\perp$ is a vector in $W^\perp$, then the orthogonal projection of $\mathbf{w}^\perp$ onto W is $\mathbf{w}^\perp$ itself.

   (i) If $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is an orthonormal basis of W, then $\mathbf{w} = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.

   (j) If V is finite-dimensional, $\mathbf{v} \in V$, and W is any subspace of V, the vector $\mathbf{w} \in W$ minimizing $||\mathbf{v} - \mathbf{w}||$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.

   (k) If $T : V \to V$ is linear, then the adjoint of T exists and is unique.

   (l) If $T : V \to V$ is linear and V is finite-dimensional, then the adjoint of T exists and is unique.

   (m) If $T : V \to F$ is linear and V is finite-dimensional, then there exists $\mathbf{w} \in V$ such that $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v} \in V$.

   (n) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(S + iT)^* = S^* + iT^*$.

   (o) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(ST)^* = S^*T^*$.

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2. For each list S of vectors in the given inner product space, apply Gram-Schmidt to calculate an orthogonal basis for $\text{span}(S)$:

   (a) $\mathbf{v}_1 = (2, 4, -4)$, $\mathbf{v}_2 = (1, -1, 4)$, $\mathbf{v}_3 = (1, 1, 1)$ in $\mathbb{R}^3$ under the standard dot product.

   (b) $\mathbf{v}_1 = (1, 2, 0, -2)$, $\mathbf{v}_2 = (1, -1, 4, 4)$, $\mathbf{v}_3 = (6, 6, 0, -9)$ in $\mathbb{R}^4$ under the standard dot product.

   (c) $\mathbf{v}_1 = x$, $\mathbf{v}_2 = x^2$, $\mathbf{v}_3 = x^3$ in $C[-1, 1]$ under the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

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3. Calculate the following things (assume any unspecified inner product is the standard one):

   (a) Write $\mathbf{v} = (-5, 5, -6)$ as a linear combination of the orthogonal basis $(i, -i, 0)$, $(1, 1, 2i)$, $(i, i, 1)$ of $\mathbb{C}^3$.

   (b) A basis for $W^\perp$, if $W = \text{span}[(1, 1, 1, 1), (2, 3, 4, 1)]$ inside $\mathbb{R}^4$.

   (c) A basis for $W^\perp$, if $W = \text{span}[(1, 1, 2i), (1, -i, 4)]$ inside $\mathbb{C}^3$. [Hint: Over $\mathbb{C}$, compute the complex conjugate of the nullspace.]

   (d) The orthogonal decomposition $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ of $\mathbf{v} = (2, 0, 11)$ into $W = \text{span}[\frac{1}{3}(1, 2, 2), \frac{1}{3}(2, -2, 1)]$ inside $\mathbb{R}^3$. Also, verify the relation $||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^\perp||^2$.

   (e) An orthogonal basis for $W = \text{span}[x, x^2, x^3]$, and the orthogonal projection of $\mathbf{v} = 1 + 2x^2$ into W, with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

   (f) The quadratic polynomial $p(x) \in P_2(\mathbb{R})$ that minimizes the expression $\int_0^1 [p(x) - \sqrt{x}]^2\,dx$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Let V be an inner product space with scalar field F. The goal of this problem is to prove the so-called "polarization identities".
   (a) If $F = \mathbb{R}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} ||\mathbf{v} + \mathbf{w}||^2 - \frac{1}{4} ||\mathbf{v} - \mathbf{w}||^2$.
   (b) If $F = \mathbb{C}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \left|\left|\mathbf{v} + i^k \mathbf{w}\right|\right|^2$.

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5. Let V be a finite-dimensional inner product space and W be a subspace of V.
   (a) Prove that $W \cap W^{\perp} = \{\mathbf{0}\}$ and deduce that $V = W \oplus W^{\perp}$. [Hint: Use $\dim(W) + \dim(W^{\perp}) = \dim(V)$.]
   (b) Let $T : V \to W$ be the function defined by setting $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ for $\mathbf{w} \in W$ and $\mathbf{w}^{\perp} \in W^{\perp}$. Prove that T is linear, that $T^2 = T$, that $\text{im}(T) = W$, and that $\ker(T) = W^{\perp}$. Conclude that T is projection onto the subspace W with kernel $W^{\perp}$.

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6. Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \to V$ is a linear transformation possessing an adjoint T^*. We say T is Hermitian (or self-adjoint) if $T = T^*$, and that T is skew-Hermitian if $T = -T^*$.
   (a) Show that T is Hermitian if and only if iT is skew-Hermitian.
   (b) Show that $T + T^*$, T^*T, and TT^* are all Hermitian, while $T - T^*$ is skew-Hermitian.
   (c) Show that T can be written as $T = S_1 + iS_2$ for unique Hermitian transformations S_1 and S_2.
   (d) Suppose T is Hermitian. Prove that $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is a real number for any vector $\mathbf{v}$.

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7. Suppose V is an inner product space over the field F (where $F = \mathbb{R}$ or $\mathbb{C}$) and $T : V \to V$ is linear. We say T is a "distance-preserving map" on V if $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v}$ in V, and we say T is an "angle-preserving map" on V if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{w} \rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.
   (a) Prove that T is distance-preserving if and only if it is angle-preserving. [Hint: Use problem 3.]

   A map $T : V \to V$ satisfying the distance- and angle-preserving conditions is called a (linear) isometry.

   (b) Show that the transformations $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S(x, y, z) = (z, -x, y)$ and $T(x, y, z) = \frac{1}{3}(x + 2y + 2z, 2x + y - 2z, 2x - 2y + z)$ are both isometries under the usual dot product.
   (c) Show that isometries are one-to-one.
   (d) Show that isometries preserve orthogonal and orthonormal sets.
   (e) Suppose T^* exists. Prove that T is an isometry if and only if T^*T is the identity transformation.
   (f) We say that a matrix $A \in M_{n \times n}(F)$ is unitary if $A^{-1} = A^*$. Show that the isometries of F^n (with its usual inner product) are precisely those maps given by left-multiplication by a unitary matrix.

   Remark: Notice that $A \in M_{n \times n}(\mathbb{C})$ is unitary if and only if the columns of A are an orthonormal basis of $\mathbb{C}$. Thus, the result of part (f) can equivalently be thought of as saying that the distance-preserving maps on $\mathbb{C}^n$ (or $\mathbb{R}^n$) are simply changes of basis from one orthonormal basis (the columns of A) to another (the standard basis).

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8. [Challenge] The goal of this problem is to give an example of an inner product space that has no orthonormal basis. Let $V = \ell^2(\mathbb{R})$ be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \ldots)$ such that $\sum_{i=1}^{\infty} a_i^2$ is finite, under componentwise addition and scalar multiplication.

   (a) Show that the pairing $\langle \{a_i\}_{i\geq 1}, \{b_i\}_{i\geq 1} \rangle = \sum_{i=1}^{\infty} a_i b_i$ is an inner product on V. (Make sure to justify why this sum converges.)

   (b) Let $\mathbf{v}_i \in V$ be the sequence with a 1 in the ith component and 0s elsewhere. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \ldots\}$ is an orthonormal set in V and that the only vector $\mathbf{w}$ orthogonal to all of the $\mathbf{v}_i$ is the zero vector. Deduce that S is a maximal orthonormal set of V that is not a basis of V.

   Part (b) shows that Gram-Schmidt does not necessarily construct an orthonormal basis of V. In fact, V has no orthonormal basis at all.

   (c) Suppose V has an orthonormal basis $\{\mathbf{e}_i\}_{i\in I}$ for some indexing set I (which is necessarily infinite), and choose a countably infinite subset $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \ldots$. Show that the sum $\mathbf{v} = \sum_{k=1}^{\infty} 2^{-k}\mathbf{e}_k$ is a well-defined vector in V that cannot be written as a (finite) linear combination of the basis $\{\mathbf{e}_i\}_{i\in I}$. [Hint: Show that $||\mathbf{v}||^2 = \lim_{n\to\infty} \left|\left|\sum_{k=1}^{n} 2^{-k}\mathbf{e}_k\right|\right|^2$ is finite.]

   Remark: The point here is that because our definition of span and basis only allows us to use finite linear combinations, these definitions are not well suited to handle infinite-dimensional spaces like $\ell^2(\mathbb{R})$. However, it is possible (by exploiting the fact that ℓ^2 is a topologically-complete metric space) to deal with these issues and define a "Schauder basis" that allows the use of infinite sums, which amounts to viewing ℓ^2 as a Hilbert space.
