

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that V is finite-dimensional over the field F , the bases α, β, γ are ordered, T is linear, and $\langle \cdot, \cdot \rangle$ is an inner product on V . Identify each of the following statements as true or false:
 - (a) If $T : V \rightarrow V$ is an isomorphism, then $[T]_{\beta}^{\beta} = Q[T]_{\alpha}^{\alpha}Q^{-1}$ where $Q = [T]_{\alpha}^{\beta}$.
 - (b) For any $T : V \rightarrow V$, there always exists an invertible matrix Q such that $[T]_{\beta}^{\beta} = Q[T]_{\alpha}^{\alpha}Q^{-1}$.
 - (c) For any $T : V \rightarrow V$, if $P = [I]_{\beta}^{\gamma}$, then it is true that $[T]_{\gamma}^{\gamma} = P[T]_{\beta}^{\beta}P^{-1}$.
 - (d) An inner product is linear in each of its components.
 - (e) There is exactly one inner product on \mathbb{R}^n .
 - (f) In any inner product space, $\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$.
 - (g) In any inner product space, $\|\mathbf{v} + \mathbf{w}\| \geq \|\mathbf{v}\| + \|\mathbf{w}\|$.
 - (h) In any inner product space, if $\langle \mathbf{v}, 2\mathbf{v} \rangle = 0$ then $\mathbf{v} = \mathbf{0}$.
 - (i) In any inner product space, if $\langle \mathbf{v}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{y} \rangle$ then $\mathbf{x} = \mathbf{y}$.
 - (j) In any inner product space, for a fixed $\mathbf{w} \in V$, the map $T : V \rightarrow F$ with $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ is linear.
 - (k) In any inner product space, for a fixed $\mathbf{w} \in V$, the map $T : V \rightarrow F$ with $T(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v} \rangle$ is linear.
 - (l) The Cauchy-Schwarz inequality holds in every inner product space.
 - (m) The triangle inequality holds in real inner product spaces but not complex inner product spaces.
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2. Suppose $V = P_3(\mathbb{R})$, with standard basis $\beta = \{1, x, x^2, x^3\}$, and let $T : V \rightarrow V$ be the linear transformation with $T(1) = 1 - x + x^2 - x^3$, $T(x) = 2x - x^3$, and $T(x^2) = 3 + x - x^3$, and $T(x^3) = 1 - x^2$.
 - (a) Find $[T]_{\beta}^{\beta}$.
 Now let γ be the ordered basis $\gamma = \{x^3, x^2, x + 1, x\}$.
 - (b) Find the change-of-basis matrix $Q = [I]_{\beta}^{\gamma}$ and its inverse.
 - (c) For $\mathbf{v} = 2 - x - 2x^2 + x^3$, compute $[\mathbf{v}]_{\beta}$, $[\mathbf{v}]_{\gamma}$, and verify that $[\mathbf{v}]_{\gamma} = Q[\mathbf{v}]_{\beta}$.
 - (d) Find $[T]_{\beta}^{\gamma}$, $[T]_{\gamma}^{\beta}$, and $[T]_{\gamma}^{\gamma}$.
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3. For each of the following pairings, determine (with brief justification) whether or not it is an inner product on the given vector space:
 - (a) The pairing $\langle A, B \rangle = \text{tr}(A + B)$ on $M_{2 \times 2}(\mathbb{R})$.
 - (b) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on \mathbb{R}^2 .
 - (c) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on \mathbb{C}^2 .
 - (d) The pairing $\langle (a, b), (c, d) \rangle = ac$ on \mathbb{R}^2 .
 - (e) The pairing $\langle f, g \rangle = \int_0^1 f'(x)g(x) dx$ on $C[0, 1]$.
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4. For each pair of vectors \mathbf{v}, \mathbf{w} in the given inner product space, compute $\langle \mathbf{v}, \mathbf{w} \rangle$, $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and $\|\mathbf{v} + \mathbf{w}\|$, and verify the Cauchy-Schwarz and triangle inequalities for \mathbf{v} and \mathbf{w} :
 - (a) $\mathbf{v} = (1, 2, 2, 4)$ and $\mathbf{w} = (4, 1, 4, 4)$ in \mathbb{R}^4 with the standard inner product.
 - (b) $\mathbf{v} = (i, -i, 1 + i)$ and $\mathbf{w} = (2 - i, 4, -2i)$ in \mathbb{C}^3 with the standard inner product.
 - (c) $\mathbf{v} = e^t$ and $\mathbf{w} = e^{2t}$ in $C[0, 1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$.
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Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose that V is a finite-dimensional vector space and $T : V \rightarrow V$ is linear.

- (a) Suppose there exists a basis β of V such that $[T]_{\beta}^{\beta}$ is a diagonal matrix whose diagonal entries are all 1s and 0s. Show that T is a projection map (i.e., that $T^2 = T$).
 - (b) Conversely, suppose that T is a projection map. Show that there exists a basis β of V such that $[T]_{\beta}^{\beta}$ is a diagonal matrix whose diagonal entries are all 1s and 0s. [Hint: As shown on homework 4, $V = \ker(T) \oplus \text{im}(T)$; take β be a basis of $\ker(T)$ followed by a basis of $\text{im}(T)$.]
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6. Let V be an inner product space.

- (a) If $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on V , show that $\langle \cdot, \cdot \rangle_3 = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ is also an inner product on V , where $\langle \mathbf{v}, \mathbf{w} \rangle_3 = \langle \mathbf{v}, \mathbf{w} \rangle_1 + \langle \mathbf{v}, \mathbf{w} \rangle_2$.
 - (b) If $\langle \cdot, \cdot \rangle_1$ is an inner product on V and c is a positive real number, show that $\langle \cdot, \cdot \rangle_3 = c \langle \cdot, \cdot \rangle_1$ is also an inner product on V , where $\langle \mathbf{v}, \mathbf{w} \rangle_3 = c \langle \mathbf{v}, \mathbf{w} \rangle_1$.
 - (c) Does the collection of inner products on V form a vector space under the natural addition and scalar multiplication described above? Explain why or why not.
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7. Prove the following inequalities:

- (a) Prove that $(a_1 + a_2 + \cdots + a_n) \cdot \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq n^2$ for any positive real numbers a_1, a_2, \dots, a_n , with equality if and only if all of the a_i are equal.
 - (b) If a, b, c, d are real numbers with $a^2 + b^2 + c^2 + d^2 \leq 5$, show that $a + 2b + 3c + 4d \leq 5\sqrt{6}$.
 - (c) Prove Nesbitt's inequality: for any positive real numbers a, b, c it is true that $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$. [Hint: Apply Cauchy-Schwarz to $(\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ and $(1/\sqrt{a+b}, 1/\sqrt{b+c}, 1/\sqrt{c+a})$.]
 - (d) Prove the following generalization of Cauchy-Schwarz: if $\langle \cdot, \cdot \rangle$ is an inner product on the vector space V , it is true that $\left[\sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{w}_j \rangle \right]^2 \leq \left[\sum_{j=1}^n \langle \mathbf{v}_j, \mathbf{v}_j \rangle \right] \cdot \left[\sum_{j=1}^n \langle \mathbf{w}_j, \mathbf{w}_j \rangle \right]$ for any vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ in V . [Hint: Apply Cauchy-Schwarz to the inner product space \tilde{V} of n -tuples of elements of V .]
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8. [Challenge] Find all continuous positive functions $f(x)$ on the interval $[0, 1]$ such that $\int_0^1 f(x) dx = 1$, $\int_0^1 x f(x) dx = \alpha$, and $\int_0^1 x^2 f(x) dx = \alpha^2$ where α is a given real number.

- Remark: This was problem A2 from the 1964 Putnam exam.
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