E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2023 ~ Homework 4, due Fri Feb 10th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that V and W are arbitrary vector spaces, not necessarily finite-dimensional, over the scalar field F. Identify each of the following statements as true or false:

   (a) If $T : V \to W$ is a linear transformation, then $T(\mathbf{0}_V) = \mathbf{0}_W$.
   (b) If $T : V \to W$ has $T(\mathbf{a} + \mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b})$ for every $\mathbf{a}, \mathbf{b} \in V$ then T is a linear transformation.
   (c) If $T : V \to W$ has $T(r\mathbf{a}) = rT(\mathbf{a})$ for every $r \in F$ and every $\mathbf{a} \in V$ then T is a linear transformation.
   (d) If T is a linear transformation such that $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$, then T is one-to-one.
   (e) If $T : V \to V$ is a one-to-one linear transformation, then $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.
   (f) If $\dim(\text{im}(T)) = \dim(W)$, then T is onto.
   (g) If $T : V \to W$ is linear and S spans V, then $T(S) = \{T(\mathbf{s}) : \mathbf{s} \in S\}$ spans W.
   (h) If $T : V \to W$ is linear and S is linearly independent in V, then $T(S)$ is a linearly independent in W.
   (i) If $T : V \to W$ is linear and S is a basis for V, then $T(S)$ is a basis for W.
   (j) For any $\mathbf{v}_1, \mathbf{v}_2 \in V$ and any $\mathbf{w}_1, \mathbf{w}_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.
   (k) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 2 and whose rank is 2.
   (l) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 4 and whose rank is 1.
   (m) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 1 and whose rank is 4.
   (n) If $T : V \to W$ is linear and for any $\mathbf{w} \in W$ there is a unique $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$, then T is an isomorphism.
   (o) If V is isomorphic to W, then $\dim(V) = \dim(W)$.

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2. Calculate the following:

   (a) If $S = \{\langle 2, 1, -1\rangle, \langle -1, 2, 3\rangle, \langle -2, 3, 5\rangle, \langle 4, 1, -3\rangle\}$, find a subset of S that is a basis for $\text{span}(S)$ in $\mathbb{R}^3$.
   (b) Extend the set $S = \{\langle 1, 2, 1, 1\rangle, \langle -1, 2, 2, 2\rangle, \langle -2, 1, 2, 2\rangle\}$ to a basis of $\mathbb{Q}^4$.

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3. For each map $T : V \to W$, determine whether or not T is a linear transformation from V to W, and if it is not, identify at least one property that fails:

   (a) $V = W = \mathbb{R}^4$, $T(a, b, c, d) = (a - b, b - c, c - d, d - a)$.
   (b) $V = W = \mathbb{R}^2$, $T(a, b) = (a, b^2)$.
   (c) $V = W = M_{2\times 2}(\mathbb{Q})$, $T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} A - A \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.
   (d) $V = W = \mathbb{C}[x]$, $T(p(x)) = p(x^2) - xp'(x)$.
   (e) $V = W = M_{4\times 4}(\mathbb{F}_2)$, $T(A) = Q^{-1}AQ$, for a fixed 4×4 matrix Q.
   (f) $V = W = M_{4\times 4}(\mathbb{R})$, $T(A) = A^{-1}QA$, for a fixed 4×4 matrix Q.

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4. For each map $T : V \to W$, (i) show that T is a linear transformation, (ii) find bases for the kernel and image of T, (iii) compute the nullity and rank of T and verify the conclusion of the nullity-rank theorem, and (iv) identify whether T is one-to-one, onto, or an isomorphism.

   (a) $T : \mathbb{Q}^2 \to \mathbb{Q}^3$ defined by $T(a, b) = \langle a + b, 2a + 2b, a + b\rangle$.
   (b) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$.
   (c) $T : P_2(\mathbb{C}) \to P_3(\mathbb{C})$ defined by $T(p) = xp(x) + p'(x)$.
   (d) $T : P_3(\mathbb{F}_3) \to P_4(\mathbb{F}_3)$ defined by $T(p) = x^3 p''(x)$. [Warning: Note that $3 = 0$ in $\mathbb{F}_3$.]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose that $T : V \to W$ is a linear transformation.
   (a) If T is onto, show that $\dim(W) \le \dim(V)$.
   (b) If T is one-to-one, show that T is an isomorphism from V to $\text{im}(T)$, and deduce that $\dim(V) \le \dim(W)$.

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6. Suppose $\dim(V) = n$ and that $T : V \to V$ is a linear transformation with $T^2 = 0$: in other words, that $T(T(\mathbf{v})) = \mathbf{0}$ for every vector $\mathbf{v} \in V$.
   (a) Show that $\text{im}(T)$ is a subspace of $\ker(T)$.
   (b) Show that $\dim(\text{im}(T)) \le n/2$.

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7. Let F be a field and let V be the vector space of infinite sequences $\{a_n\}_{n\ge 1} = (a_1, a_2, a_3, a_4, \ldots)$ of elements of F. Define the left-shift operator $L : V \to V$ via $L(a_1, a_2, a_3, a_4, \ldots) = (a_2, a_3, a_4, a_5, \ldots)$ and the right-shift operator $R : V \to V$ via $R(a_1, a_2, a_3, a_4, \ldots) = (0, a_1, a_2, a_3, \ldots)$.
   (a) Show that L is a linear transformation that is onto but not one-to-one.
   (b) Show that R is a linear transformation that is one-to-one but not onto.
   (c) Deduce that on infinite-dimensional vector spaces, the conditions of being one-to-one, being onto, and being an isomorphism are not in general equivalent.
   (d) Verify that $L \circ R$ is the identity map on V, but that $R \circ L$ is not the identity map on V.
   (e) Deduce that on infinite-dimensional vector spaces, a linear transformation with a left inverse or a right inverse need not have a two-sided inverse.

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8. A linear transformation $T : V \to V$ such that $T^2 = T$ is called a projection map. The goal of this problem is to give some other descriptions of projection maps.
   (a) Suppose that $T : V \to V$ has the property that there exists a subspace W such that $\text{im}(T) = W$ and T is the identity map when restricted to W. Show that T is a projection map (it is called the projection onto the subspace W).
   (b) Conversely, suppose T is a projection map. Show that T is a projection onto the subspace $W = \text{im}(T)$.
   (c) Suppose that T is a projection map. Prove that $V = \ker(T) \oplus \text{im}(T)$. [Hint: Try writing $\mathbf{v} = [\mathbf{v} - T(\mathbf{v})] + T(\mathbf{v})$.]

   - Remark: Projection maps are so named because they represent the geometric idea of projection. For example, in the event that $W = \text{im}(T)$ is one-dimensional, the corresponding projection map T represents projecting onto that line.

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9. [Challenge] The goal of this problem is to demonstrate some of the bizarre things one can do with infinite bases.
   (a) Show that $\dim_{\mathbb{Q}} \mathbb{R} = \dim_{\mathbb{Q}} \mathbb{C}$. Deduce that there exists a $\mathbb{Q}$-vector space isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$. [Hint: Use the fact that finite-dimensional $\mathbb{Q}$-vector spaces are countable.]
   We will now use this isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$ to define a different vector space structure on $\mathbb{C}$. Intuitively, the idea is to start with the set $\mathbb{R}$ as a vector space over itself, and then use the isomorphism φ^{-1} to relabel the vectors as complex numbers, but keep the scalars as real numbers.
   (b) Let V be the set of complex numbers with the addition operation $z_1 \oplus z_2 = z_1 + z_2$ and scalar multiplication defined as follows: for $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$, set $\alpha \odot z = \varphi^{-1}[\alpha\varphi(z)]$. Show that $(V, \oplus, \odot)$ is an $\mathbb{R}$-vector space.
   (c) Using the vector space structure defined in (b), show that $\dim_{\mathbb{R}} V = 1$.

   - Remark: The point of (c) is that by changing the definition of scalar multiplication, we can make $\mathbb{C}$ into a 1-dimensional $\mathbb{R}$-vector space. By doing a similar thing in the reverse order, we could even make $\mathbb{R}$ into a 2-dimensional $\mathbb{C}$-vector space.
