E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2023 ~ Homework 3, due Fri Feb 3rd

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:
   (a) The zero vector space has no basis.
   (b) The set $\{\mathbf{0}\}$ is a basis for the zero vector space.
   (c) Every vector space has a finite basis.
   (d) Every vector space has a unique basis.
   (e) No vector space has a unique basis.
   (f) Every subspace of a finite-dimensional vector space is finite-dimensional.
   (g) Every subspace of an infinite-dimensional vector space is infinite-dimensional.
   (h) If $V = M_{m\times n}(F)$, then $\dim_F V = mn$.
   (i) If $V = F[x]$, then $\dim_F V$ is undefined.
   (j) If $V = P_n(F)$, then $\dim_F V = n$.
   (k) If $\dim(V) = 5$, then there exists a set of 5 vectors in V that span V but are not linearly independent.
   (l) If $\dim(V) = 5$, then a set of 4 vectors in V cannot span V.
   (m) If $\dim(V) = 5$, then a set of 4 vectors in V cannot be linearly independent.
   (n) If $\dim(V) = 5$, then there is a unique subspace of V of dimension 0.
   (o) If $\dim(V) = 5$, then there is a unique subspace of V of dimension 1.
   (p) If $\dim(V) = 5$, then there is a unique subspace of V of dimension 5.
   (q) If V is infinite-dimensional, then any infinite linearly-independent subset is a basis.

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2. Find a basis for, and the dimension of, each of the following vector spaces:
   (a) The space of 3×3 symmetric matrices over $F = \mathbb{C}$.
   (b) The row space, column space, and nullspace of $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$ over $\mathbb{R}$.
   (c) The vectors in $\mathbb{Q}^5$ of the form $\langle a, b, c, d, e\rangle$ with $e = a + b$ and $b = c = d$, over $\mathbb{Q}$.
   (d) The row space, column space, and nullspace of $M = \begin{bmatrix} 1 & 3 & -2 & -6 & 8 \\ 2 & -1 & 2 & 8 & 1 \\ -1 & 1 & 1 & -3 & 3 \end{bmatrix}$ over $\mathbb{C}$.
   (e) The polynomials $p(x)$ in $P_4(\mathbb{R})$ such that $p(1) = 0$.
   (f) The matrices A in $M_{2\times 2}(\mathbb{R})$ such that $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Let W be a vector space. Recall that if A and B are two subspaces of W then their sum is the set $A + B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}$.

   (a) Suppose that $A \cap B = \{\mathbf{0}\}$. If α is a basis for A and β is a basis for B, prove that α and β are disjoint and that $\alpha \cup \beta$ is a basis for $A + B$.

   (b) Now suppose that α is a basis for A and β is a basis for B. If $\alpha \cup \beta$ is a basis for $A + B$ and α and β are disjoint, prove that $A \cap B = \{\mathbf{0}\}$.

   The situation in (a)-(b) is very important and arises often. Explicitly, if A and B are two subspaces of W such that $A + B = W$ and $A \cap B = \{\mathbf{0}\}$ is the trivial subspace, we write $W = A \oplus B$ and call W the (internal) direct sum of A and B. (The idea is that we may "decompose" W into two independent pieces A and B.)

   (c) Show that $\mathbb{R}^2$ is the direct sum of the subspaces given by the x-axis and the y-axis, and is also the direct sum of the subspaces given by the x-axis and the line $y = 3x$.

   (d) Prove that $W = A \oplus B$ if and only if every vector $\mathbf{w} \in W$ can be written uniquely in the form $\mathbf{w} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a} \in A$ and $\mathbf{b} \in B$.

   (e) If $W = A \oplus B$, show that $\dim(W) = \dim(A) + \dim(B)$. Show using an explicit counterexample that the converse statement need not hold.

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4. Let V be a vector space such that $\dim_{\mathbb{C}} V = n$. Prove that if V is now considered a vector space over $\mathbb{R}$ (using the same addition and scalar multiplication), then $\dim_{\mathbb{R}} V = 2n$.

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5. Let F be a finite field of cardinality q. The goal of this problem is to compute the number of invertible $n \times n$ matrices in $M_{n\times n}(F)$.

   (a) Suppose W is a k-dimensional subspace of F^n. Show that W contains exactly q^k vectors.

   (b) Show that the number of invertible $n \times n$ matrices in $M_{n\times n}(F)$ is equal to the number of ordered lists $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of n linearly independent vectors from F^n.

   (c) For any integer $0 \le k \le n$, show that there are exactly $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$ ordered lists $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of k linearly independent vectors from F^n. [Hint: Count the number of ways to choose the vector $\mathbf{v}_{k+1}$ not in $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.]

   (d) Deduce that the number of invertible $n \times n$ matrices in $M_{n\times n}(F)$ is equal to $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}) \cdots (q^n - q^{n-1})$. In particular, find the number of invertible 5×5 matrices over the field $\mathbb{F}_2$.

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6. [Challenge] Zorn's lemma states that if $\mathcal{F}$ is a nonempty partially-ordered set in which every chain has an upper bound (i.e., an element $U \in \mathcal{F}$ such that $X \le U$ for all X in the chain), then $\mathcal{F}$ contains a maximal element (i.e., an element $M \in \mathcal{F}$ such that if $M \le Y$ for some $Y \in \mathcal{F}$, then in fact $Y = M$). The goal of this problem is to use Zorn's lemma to prove that any linearly independent set can be extended to a basis.

   (a) Suppose that S is a maximal linearly-independent subset of a vector space V (this means that if T is any linearly-independent subset of V containing S, then in fact $T = S$). Prove that S is a basis of V.

   (b) Suppose $\mathcal{C}$ is a chain of linearly independent subsets of V (i.e., a collection of linearly independent subsets with the property that $A \subseteq B$ or $B \subseteq A$ for any $A, B \in \mathcal{C}$). Show that $U = \bigcup_{A\in\mathcal{C}} A$ is also linearly independent. [Hint: A linear dependence can only involve finitely many vectors.]

   (c) Prove that every linearly independent subset of V can be extended to a basis.
