E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2023 ~ Homework 11, due Fri Apr 14th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $\Phi : V \times V \to F$ be a bilinear form. Identify each of the following statements as true or false:
   (a) If Φ is a symmetric bilinear form, then $[\Phi]_\beta$ is a symmetric matrix for any basis β.
   (b) If $[\Phi]_\beta$ is a symmetric matrix for some basis β, then Φ is a symmetric bilinear form.
   (c) If $\mathcal{B}(V)$ is the space of all bilinear forms on V and $\dim_F(V) = n$, then $\dim_F \mathcal{B}(V) = 2n$.
   (d) Congruent matrices have the same eigenvalues.
   (e) Congruent matrices have the same eigenvectors.
   (f) Every $n \times n$ symmetric matrix over $\mathbb{R}$ is congruent to a diagonal matrix.
   (g) Every $n \times n$ symmetric matrix over an arbitrary field F is congruent to a diagonal matrix.
   (h) The function $Q(x, y) = xy$ on $\mathbb{R}^2$ is a quadratic form.
   (i) The function $Q(x, y, z) = x^2 - 4xy + xyz + z^2$ on $\mathbb{R}^3$ is a quadratic form.
   (j) The function $Q(f) = \int_0^1 x\, f(x)^2\, dx$ on $\mathbb{R}[x]$ is a quadratic form.
   (k) Every quadratic form over $\mathbb{R}$ is a bilinear form.
   (l) Every quadratic form over an arbitrary field is a bilinear form.
   (m) The second derivatives test classifies any critical point as a local minimum, local maximum, or saddle.
   (n) If both eigenvalues of the 2×2 real symmetric matrix S are positive, then the graph of $(x, y)\cdot S\cdot(x, y)^T = 1$ in $\mathbb{R}^2$ will be an ellipse.
   (o) If one eigenvalue of the 2×2 real symmetric matrix S is zero and the other is nonzero, then the graph of $(x, y) \cdot S \cdot (x, y)^T = 1$ in $\mathbb{R}^2$ will be a hyperbola.

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2. For each symmetric matrix S over each given field, find an invertible matrix Q and diagonal matrix D such that $Q^T SQ = D$:
   (a) $S = \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix}$ over $\mathbb{Q}$.
   (b) $S = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 6 \\ -2 & 6 & 7 \end{bmatrix}$ over $\mathbb{Q}$.
   (c) $S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ over $\mathbb{Q}$.

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3. Consider the bilinear form $\Phi[(a, b), (c, d)] = 4ac - 2ad - 2bc + 7bd$ on $\mathbb{R}^2$ with associated quadratic form Q.
   (a) Write down Q explicitly and also find $[\Phi]_\beta$ for $\beta = \{(1, 0), (0, 1)\}$.
   (b) Find an orthonormal basis γ for $\mathbb{R}^2$ such that $[\Phi]_\gamma$ is diagonal, and compute the diagonalization $[\Phi]_\gamma$.
   (c) Describe the shape of the quadratic variety $Q(x, y) = 1$ in $\mathbb{R}^2$ as one of the 3 standard conic sections.
   (d) Classify the critical point of $Q(x, y)$ at $(0, 0)$ as a local minimum, local maximum, or saddle point.
   (e) Calculate the signature and index of Q, and determine the definiteness of Q.

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4. Consider the quadratic form $Q(x, y, z) = 11x^2 + 40xy - 16xz - 16y^2 - 16yz + 5z^2$ on $\mathbb{R}^3$.

   (a) Find the symmetric matrix S associated to the underlying bilinear form for Q with respect to the standard basis $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

   (b) Give an explicit orthonormal change of basis that diagonalizes Q, and find the resulting diagonalization.

   (c) Describe the shape of the quadratic variety $Q(x, y, z) = 1$ in $\mathbb{R}^3$ as one of the 9 standard quadric surfaces.

   (d) Classify the critical point of $Q(x, y, z)$ at $(0, 0, 0)$ as a local minimum, local maximum, or saddle point.

   (e) Calculate the signature and index of Q, and determine the definiteness of Q.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. For $A, B \in M_{n\times n}(F)$, recall that we say A is congruent to B when there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^T AQ$. Prove that congruence is an equivalence relation on $M_{n\times n}(F)$.

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6. Suppose $T : V \to V$ is a linear operator on the real inner product space V with inner product $\langle \cdot, \cdot \rangle$. Define the map $\Phi : V \times V \to F$ by setting $\Phi(\mathbf{v}, \mathbf{w}) = \langle T(\mathbf{v}), \mathbf{w} \rangle$.

   (a) Show that Φ is a bilinear form on V.

   (b) Show that Φ is symmetric if and only if T is Hermitian.

   (c) If V is finite-dimensional, prove that Φ is an inner product on V if and only if T is positive-definite and Hermitian. [Hint: Show that [I3] requires all eigenvalues of T to be positive.]

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7. In multivariable calculus, the following more explicit version of the second derivative test is often taught[1]:

   - Theorem (Second Derivatives Test in $\mathbb{R}^2$): Suppose P is a critical point of $f(x, y)$, and let D be the value of the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at P. If $D > 0$ and $f_{xx} > 0$, then the critical point is a minimum. If $D > 0$ and $f_{xx} < 0$, then the critical point is a maximum. If $D < 0$, then the critical point is a saddle point. If $D = 0$, then the test is inconclusive.

   Using our general version of the second derivatives test, prove this variation. [Hint: Note that $D = \det(H) = \lambda_1\lambda_2$; then examine what information the sign of D yields about the eigenvalues λ_1, λ_2.]

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8. [Challenge] Let A be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The goal of this problem is to prove the Courant-Fisher theorem: that $\lambda_i = \min_{\dim W=i-1} \max_{||\mathbf{v}||=1,\mathbf{v}\in W^\perp}(\mathbf{v}^*A\mathbf{v})$ for each $1 \leq i \leq n$. This characterization of the eigenvalues in terms of a min-max property is useful in practical computations, particularly the $i = 1$ case: $\lambda_1 = \max_{||\mathbf{v}||=1}(\mathbf{v}^*A\mathbf{v})$.

   (a) Show that it suffices to prove the Courant-Fisher theorem when the matrix A is diagonal.

   Per (a), we now assume that A is diagonal and that for $\mathbf{v} = (x_1, \ldots, x_n)$ we have $\mathbf{v}^*A\mathbf{v} = \lambda_1x_1^2 + \cdots + \lambda_nx_n^2$.

   (b) Show that $\lambda_i \geq \min_{\dim W=i-1} \max_{||\mathbf{v}||=1,\mathbf{v}\in W^\perp}(\mathbf{v}^*A\mathbf{v})$. [Hint: Take W to be the subspace spanned by the first $i - 1$ coordinate vectors.]

   (c) Prove that $\lambda_i \leq \min_{\dim W=i-1} \max_{||\mathbf{v}||=1,\mathbf{v}\in W^\perp}(\mathbf{v}^*A\mathbf{v})$. [Hint: For any W of dimension $i - 1$, let V_i be the subspace spanned by the first i coordinate vectors and take $\mathbf{v} \in V_i \cap W$.]

   (d) Deduce that $\lambda_i = \min_{\dim W=i-1} \max_{||\mathbf{v}||=1,\mathbf{v}\in W^\perp}(\mathbf{v}^*A\mathbf{v})$ for each i.

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[1] The statement of this theorem is copied directly from my multivariable calculus course notes, in fact!