E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2023 ~ Homework 10, due Fri Apr 7th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false:
   (a) Every real Hermitian matrix is diagonalizable.
   (b) Every real symmetric matrix is diagonalizable.
   (c) Every complex Hermitian matrix is diagonalizable.
   (d) Every complex symmetric matrix is diagonalizable.
   (e) If $A \in M_{k\times k}(\mathbb{R})$ has all columns sum to 1, then 1 is an eigenvalue of A.
   (f) If $A \in M_{k\times k}(\mathbb{R})$ has nonnegative entries and all columns sum to 1, then $\lim_{n\to\infty} A^n$ exists.
   (g) If $A \in M_{k\times k}(\mathbb{R})$ has positive entries and all columns sum to 1, then $\lim_{n\to\infty} A^n$ exists.
   (h) If $V = \mathbb{R}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$ is the usual inner product on $\mathbb{R}^2$, then Φ is a bilinear form on V.
   (i) If $V = \mathbb{C}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v}\cdot\overline{\mathbf{w}}$ is the usual inner product on $\mathbb{C}^2$, then Φ is a bilinear form on V.
   (j) If $V = \mathbb{R}$ and $\Phi(x, y) = x + 2y$, then Φ is a bilinear form on V.
   (k) If $V = F^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \det(\mathbf{v}, \mathbf{w})$, the determinant of the matrix with columns $\mathbf{v}$,$\mathbf{w}$, then Φ is a bilinear form on V.
   (l) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \mathrm{tr}(AB)$, then Φ is a bilinear form on V.
   (m) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \det(AB)$, then Φ is a bilinear form on V.
   (n) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 x f(x)g(x)\,dx$, then Φ is a bilinear form on V.
   (o) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 f'(x)g'(x)\,dx$, then Φ is a bilinear form on V.

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2. Solve the following problems:
   (a) Find a formula for the nth power of the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$.
   (b) In Diagonalizistan there are two cities: City A and City B. Each year, 2/5 of the residents of City A move to City B, and 2/3 of the residents of City B move to City A; the remaining residents stay in their current city. If in year 0 the populations of Cities A and B are 2000 and 6000 residents respectively, find the populations of the two cities in year n and determine what happens as $n \to \infty$.

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3. Solve each system of differential equations:
   (a) Find the general solution to $y_1' = 7y_1 + y_2$ and $y_2' = 9y_1 - y_2$.
   (b) Find the general solution to $y_1' = 3y_1 - 2y_2$ and $y_2' = y_1 + y_2$.
   (c) Find the general solution to $y'' - 4y = 0$. [Hint: Set $z = y'$ and convert to a system of linear equations.]
   (d) Find the general solution to $y_1' = 2y_2 + \sec(2x)$ and $y_2' = -2y_1$.
   (e) Solve the system $\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}$, where $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

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4. For each bilinear form on each given vector space, compute $[\Phi]_\beta$ for the given basis β:
   (a) The pairing $\Phi((a, b, c), (d, e, f)) = ad + ae - 2be + 3cd + cf$ on $V = F^3$ with β the standard basis.
   (b) The pairing $\Phi(p, q) = p(-1)q(2)$ on $V = P_3(\mathbb{R})$ with $\beta = \{1, x, x^2, x^3\}$.
   (c) The pairing $\Phi(A, B) = \mathrm{tr}(AB)$ on $V = M_{2\times 2}(\mathbb{C})$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. The goal of this problem is to give two proofs of Binet's formula for the Fibonacci-Virahanka numbers defined by the recurrence $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$; the next few terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Explicitly, for $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \dfrac{1-\sqrt{5}}{2}$, Binet's formula says that $F_n = \dfrac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$.

   (a) Show that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and deduce that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

   (b) Find a formula for the nth power of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and use the result to deduce Binet's formula.

   (c) Let W be the space of all real sequences $\{a_n\}_{n\geq 0}$ such that $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$. Show that W is a 2-dimensional vector space over $\mathbb{R}$.

   (d) With notation as in (c), show that the sequences $\{\varphi^n\}_{n\geq 0}$ and $\{\overline{\varphi}^n\}_{n\geq 0}$ are a basis for W. Deduce that there exist constants C and D such that $F_n = C\varphi^n + D\overline{\varphi}^n$ and then deduce Binet's formula.

   Remark: Both of these methods extend generally to solve general linear recurrences of the form $a_{n+1} = C_1 a_n + C_2 a_{n-2} + \cdots + C_k a_{n-k}$ for constants $C_1, \ldots, C_k$. Additionally, the matrix formula in (a) is a good source of other Fibonacci identities.

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6. Suppose V is finite-dimensional with scalar field F and $T : V \to V$ is linear. We say the polynomial $q(x) \in F[x]$ annihilates T if $q(T) = 0$.

   (a) Show that the set of polynomials in $F[x]$ annihilating T is a vector space.

   We define the minimal polynomial of T to be the monic polynomial $m(t) \in F[t]$ of smallest positive degree annihilating T. For example, the minimal polynomial of the identity transformation is $m(t) = t - 1$.

   (b) Show that every polynomial that annihilates T is divisible by the minimal polynomial. [Hint: Use polynomial division.]

   (c) Conclude that the minimal polynomial divides the characteristic polynomial.

   (d) Suppose λ is an eigenvalue of T. Prove that λ is a root of the minimal polynomial of T, and deduce that the minimal polynomial and the characteristic polynomial have the same set of roots. [Hint: Consider the Jordan form of an associated matrix A.]

   (e) Parts (c) and (d) gives a moderately effective way to find the minimal polynomial, namely, test divisors of the characteristic polynomial that have all of the same roots (though not necessarily the same multiplicities). Using this method or otherwise, find the minimal polynomials of the matrices $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$.

   (f) Show that similar matrices have the same minimal polynomial.

   (g) Show that the minimal polynomial of the $k \times k$ Jordan block with eigenvalue λ is $m(t) = (t - \lambda)^k$.

   (h) Show that the exponent of $t - \lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ in the Jordan canonical form of A.

   (i) Show that a matrix is diagonalizable over $\mathbb{C}$ if and only if its minimal polynomial has no repeated roots.

   (j) Show that the minimal polynomial of a 2×2 matrix uniquely determines its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 2×2 matrices with minimal polynomials $m(t) = t^2 + t$, $t^2 + 1$, and $t - 3$ over $\mathbb{C}$.

   (k) Show the minimal and characteristic polynomials of a 3×3 matrix together uniquely determine its Jordan canonical form. Illustrate by finding the Jordan canonical forms of the 3×3 matrices with $(m(t), p(t))$ equal to (t, t^3), (t^2, t^3), (t^3, t^3), $(t^2 - t, t^3 - t^2)$, $(t^2 - t, t^3 - 2t^2 + t)$.

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7. [Challenge] The goal of this problem is to characterize when the limit of matrix powers $\lim_{n\to\infty} A^n$ converges. Suppose J is a $d \times d$ Jordan block matrix with eigenvalue $\lambda \in \mathbb{C}$ and let $N = J - \lambda I_d$ be the matrix with 1s directly above the diagonal and 0s elsewhere.

   (a) Show that $J^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{d}\lambda^{n-d}N^d$ for each $n \geq 1$.

   (b) Show that $\lim_{n\to\infty} J^n$ exists if and only if $|\lambda| < 1$ or if $\lambda = 1$ and $d = 1$.

   (c) Let A be a square complex matrix. Show that $\lim_{n\to\infty} A^n$ exists if and only if 1 is the only eigenvalue of A of absolute value ≥ 1 and the dimension of the 1-eigenspace equals its multiplicity as a root of the characteristic polynomial.

   (d) Suppose M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1) such that some power of M has all positive entries. Show that $\lim_{n\to\infty} M^n$ converges to a matrix whose columns are all 1-eigenvectors of M. [Hint: Use the results of the challenge problem from homework 9 applied to an appropriate power of M.]
