
Directions: Read ALL of the following directions.

This is an open-notes, open-homework, open-textbook exam. There is no official time limit, but it is suggested that you should be able to solve most of the problems within approximately 6 hours.

There are 17 problems on this exam, with points as indicated.

Justify any answers you give, including computations. You may freely refer to results from in class, from the course notes, course assignments and solutions, and the course textbook, but please make it clear what results you are using.

Proofs and explanations are expected to be clear, concise, and correct.

In problems with multiple parts, you MAY use the results of previous parts in later parts, even if you were unable to solve the earlier parts correctly.

You MAY use a computer for typesetting and to access any material on the course webpage (e.g., the course notes and homework solutions), and to perform computations. Any such computations must be clearly identified and justified as correct.

You MAY NOT use a computer to access any other information.

You MAY ask the instructor for help on any part of the exam, during office hours or via email, and in fact YOU ARE ENCOURAGED to do so.

You MAY NOT discuss the material on this exam with anyone except for the instructor (until after the due date). This includes asking others for hints or solutions, searching for information about the problems online, or posting about the problems on discussion forums.

Are we going to have to think today, or is it going to be all math?

(Anonymous Student)

Arithmetic! Algebra! Geometry! Grandiose trinity! Whoever has not known you is without sense!

Comte de Lautreamont

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

John von Neumann

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1. (10) Identify each of the following statements as true or false:

   (a) If W is a subspace of the vector space V, then there exists a basis for V containing a basis for W.

   (b) There exists a linear $T : \mathbb{R}^5 \to M_{2\times 2}(\mathbb{R})$ whose nullity and rank are equal.

   (c) If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then $||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$.

   (d) The matrices $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ are similar over $\mathbb{C}$.

   (e) If $A \in M_{6\times 6}(\mathbb{R})$ has $\det(tI_6 - A) = t^6 - t^2$, then the generalized 0-eigenspace of A has dimension 2.

   (f) The pairing $\Phi(A, B) = \mathrm{tr}(AB)$ is a bilinear form on $M_{5\times 5}(\mathbb{R})$.

   (g) Every symmetric real matrix is congruent over the real numbers to a diagonal matrix.

   (h) A quadratic form on $\mathbb{R}^8$ with index 4 and signature 4 is positive semidefinite.

   (i) Every matrix in $M_{n\times n}(\mathbb{C})$ is the product of a unitary matrix, a diagonal matrix, and a unitary matrix.

   (j) For any $A \in M_{m\times n}(\mathbb{R})$ and $\mathbf{c} \in \mathbb{R}^m$, it is true that $\left|\left|AA^\dagger \mathbf{c} - \mathbf{c}\right|\right| \le ||A\mathbf{y} - \mathbf{c}||$ for any $\mathbf{y} \in \mathbb{R}^n$.

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2. (6) Consider the derivative map $D : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $D[f(x)] = f'(x)$.

   (a) Find the matrix $[D]_\beta^\beta$ of D with respect to the basis $\beta = \{1, x, x^2, x^3\}$.

   (b) Find the eigenvalues and a basis for each eigenspace of D.

   (c) Is D diagonalizable? What is its Jordan canonical form?

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3. (6) Suppose the characteristic polynomial of the 7×7 matrix A is $p(t) = (t+1)^4(t-2)^2(t-3)$.

   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.

   (b) List all possible Jordan canonical forms of A up to equivalence.

   (c) Find the Jordan canonical form of A if $\mathrm{rank}(I + A) = 4$ and there are two linearly independent 2-eigenvectors of A.

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4. (8) Let $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$.

   (a) Find the eigenvalues of A and a basis for each eigenspace.

   (b) Find a real orthogonal matrix Q (i.e., with $Q^T = Q^{-1}$) such that $Q^{-1}AQ$ is diagonal.

   (c) Find a formula for the nth power A^n.

   (d) Solve the system of differential equations $y_1' = 3y_1 + 4y_2$, $y_2' = 4y_1 - 3y_2$ using any method.

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5. (6) Consider the symmetric bilinear form $\Phi[(a,b,c),(d,e,f)] = 6ad - 2ae - 2af - 2bd + 5be - 2cd + 7cf$ on $\mathbb{R}^3$ and let $Q(x,y,z)$ be the associated quadratic form.

   (a) Find the matrix $S = [\Phi]_\beta$ with respect to the standard basis $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$.

   (b) Evaluate $Q(x,y,z)$ explicitly.

   (c) Give an explicit orthonormal basis γ of $\mathbb{R}^3$ such that the associated matrix $[\Phi]_\gamma$ is diagonal.

   (d) Identify the surface $Q(x,y,z) = 1$ in $\mathbb{R}^3$ as one of the 9 standard quadric surfaces.

   (e) Classify the critical point of $Q(x,y,z)$ at $(0,0,0)$ as a local minimum, local maximum, or saddle point.

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6. (6) If W_1 and W_2 are subspaces of V, show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. [Hint: If not, consider the sum of a vector in W_2 not in W_1 with a vector in W_1 not in W_2.]

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7. (6) Suppose $T : V \to W$ is a linear transformation where V and W are finite-dimensional and $\dim(V) > \dim(W)$. Prove that there must exist vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ but $\mathbf{v}_1 \neq \mathbf{v}_2$.

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8. (6) Show $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ for any subspaces W_1, W_2 of a finite-dimensional inner product space V.

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9. (9) Suppose V is an inner product space and $T : V \to V$ is linear. Recall that we say T is an isometry if $\langle T(\mathbf{v}), T(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$.
   (a) If T is an isometry, show that any eigenvalue $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$.
   (b) Conversely, suppose that V possesses an orthonormal basis of eigenvectors for T each of whose eigenvalues λ has $|\lambda| = 1$. Prove that T is an isometry. [Hint: Compute $\langle T(\mathbf{v}), T(\mathbf{w})\rangle$ in terms of this basis.]
   (c) If V is finite-dimensional and T is an isometry that is also Hermitian, show that T^2 is the identity map and that T is unitary. [Hint: Consider the diagonalization.]

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10. (6) Suppose V is finite-dimensional and $T : V \to V$ is a linear operator on V whose characteristic polynomial splits completely (i.e., so that all eigenvalues of T lie in the scalar field of V).
    (a) Show that T is diagonalizable if and only if $\ker(T - \lambda I) = \ker(T - \lambda I)^2$ for all eigenvalues λ of T.
    (b) Show that T is diagonalizable if and only if $T - \lambda I$ and $(T - \lambda I)^2$ have the same rank for all eigenvalues λ of T.

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11. (10) A matrix $A \in M_{n\times n}(\mathbb{C})$ is called nilpotent if A^k is the zero matrix for some positive integer k.
    (a) If A is diagonalizable and nilpotent, show that it is the zero matrix.
    (b) If A is nilpotent, show that the only eigenvalue of A is $\lambda = 0$.
    (c) Conversely, if the only eigenvalue of A is $\lambda = 0$, show that A is nilpotent. [Hint: Consider the characteristic polynomial.]
    (d) Find all possible Jordan canonical forms for a 4×4 nilpotent matrix.

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12. (6) Suppose $A \in M_{n\times n}(\mathbb{C})$. Prove that $\det(e^A) = e^{\operatorname{tr}(A)}$, and deduce that e^A is always invertible. [Hint: Put A in Jordan form.]

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13. (8) Let V be a (not necessarily finite dimensional) vector space with scalar field F, with $c \in F$, and suppose that Φ_1 and Φ_2 are two bilinear forms on V.
    (a) Show that $\Phi_1 + c\Phi_2$ is a bilinear form on V. Note that $\Phi_1 + c\Phi_2$ is defined pointwise, so $(\Phi_1 + c\Phi_2)(\mathbf{v}, \mathbf{w}) = \Phi_1(\mathbf{v}, \mathbf{w}) + c\Phi_2(\mathbf{v}, \mathbf{w})$.
    (b) Show that $(\Phi_1 + c\Phi_2)^T = \Phi_1^T + c\Phi_2^T$. Deduce that if Φ_1 and Φ_2 are symmetric then so is $\Phi_1 + c\Phi_2$.
    (c) Show that the set of all bilinear forms on V is a vector space, and that the space of symmetric bilinear forms is a subspace of it.

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14. (8) Let A, B be elements of $M_{n\times n}(F)$ for an arbitrary field F.
    (a) If A and B are similar over F, prove that A^T and B^T are also similar over F.
    (b) If A and B are congruent over F, prove that A^T and B^T are also congruent over F.
    (c) If A and B are similar over F and invertible, prove that A^{-1} and B^{-1} are also similar over F.
    (d) If A is invertible and diagonalizable over F, prove that A^{-1} is also diagonalizable over F.

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15. (6) Suppose $A \in M_{m\times n}(\mathbb{C})$. Show that $I_n + A^*A$ is positive definite. [Hint: Show A^*A is positive-semidefinite.]

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16. (6) Suppose $A \in M_{m\times n}(\mathbb{C})$. If W is an $m \times m$ unitary matrix, show that the pseudoinverse $(WA)^\dagger = A^\dagger W^*$.

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17. (12) Prove or disprove each of the following statements:
    (a) "Linear algebra has no practical applications."
    (b) "Linear algebra is a boring and useless subject."
