E. Dummit's Math 3527 ~ Number Theory I, Spring 2023 ~ Homework 2, due Tue Jan 24th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Find the following:
   (a) The gcd and lcm of 288 and 600.
   (b) The gcd and lcm of $2^8 3^{11} 5^7 7^8 11^2$ and $2^4 3^8 5^7 7^7 11^{11}$.
   (c) The prime factorizations of 2022 and 2023.
   (d) The prime factorizations of $2022^{2023}$ and $2023^{2022}$.
   (e) The prime factorizations of 111, 1001, and 111111.
   (f) A positive integer $n$ such that $n/2$ is a perfect square and $n/3$ is a perfect cube.

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2. It is sometimes claimed (occasionally in actual textbooks) that if $p_1, p_2, \dots, p_k$ are the first $k$ primes, then the number $n = p_1 p_2 \cdots p_k + 1$ used in Euclid's proof is always prime for any $k \geq 1$. Find a counterexample to this statement (make sure to justify that it is actually a counterexample).

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Suppose that $n$ is a positive integer with prime factorization $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for distinct primes $p_1, \dots, p_k$.
   (a) Show that the number of positive integers $d$ dividing $n$ is equal to $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$. [Hint: Consider the prime factorization of $d$.]
   (b) Prove that a positive integer $n$ has an odd number of divisors if and only if $n$ is a perfect square.
   (c) Show that the sum of the positive divisors of $n$, denoted $\sigma(n)$, is equal to $(1 + p_1 + \cdots + p_1^{a_1})(1 + p_2 + \cdots + p_2^{a_2}) \cdots (1 + p_k + \cdots + p_k^{a_k})$. [Hint: Try distributing out the product.]
   (d) A perfect number is a positive integer $N$ such that $\sigma(N) = 2N$ (typically phrased as "the sum of all of the proper divisors of $n$ equals $n$ itself). Show that if $2^n - 1$ is a prime number, then the number $N = 2^{n-1}(2^n - 1)$ is perfect.
   (e) Show that 28, 496, 8128 are perfect numbers.

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4. The goal of this problem is to study which numbers of the form $N = a^k - 1$ can be prime, where $a$ and $k$ are positive integers greater than 1.
   (a) Show that $n^k - 1$ is divisible by $n - 1$, for any integer $n$.
   (b) Show that if $a > 2$, then $N = a^k - 1$ is not prime.
   (c) Show that if $k$ is composite, then $N = 2^k - 1$ is not prime. [Hint: If $k = rs$, show $N$ is divisible by $2^r - 1$.]
   (d) Show that the only primes of the form $N = a^k - 1$ are those of the form $2^p - 1$ where $p$ is a prime. (Such primes are called Mersenne primes.) Are all the numbers of the form $2^p - 1$ ($p$ prime) necessarily prime?

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5. Prove that $\log_3 7$ is irrational. [Hint: Suppose otherwise, so that $\log_3 7 = a/b$. Convert this to statement about positive integers and find a contradiction.]

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6. Let $n$ be a positive integer greater than 1.
   (a) Show that if $n$ is composite, then $n$ must have at least one divisor $d$ with $d \leq \sqrt{n}$. Deduce that if $n$ is composite, then $n$ has at least one prime divisor $p \leq \sqrt{n}$. [Hint: Write $n = ab$ where $1 < a \leq b < n$.]
   (b) Show that if no prime less than or equal to $\sqrt{n}$ divides $n$, then $n$ is prime.
   (c) Show explicitly that $n = 109$ and $n = 251$ are prime by verifying that they are not divisible by any prime $\leq \sqrt{n}$.

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