E. Dummit's Math 3527 $\sim$ Number Theory I, Spring 2023 $\sim$ Homework 1, due Tue Jan 17th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Each item below contains a proposition and a claimed proof of the proposition. Each proof has an error of some kind: identify the error and briefly explain why it causes the proof to be incorrect.

   (a) Proposition: If a and b are integers with $a = b$, then $a = 0$.
   Proof: Suppose $a = b$. Multiplying by a yields $a^2 = ab$ and then subtracting b^2 yields $a^2 - b^2 = ab - b^2$. Factoring yields $(a-b)(a+b) = (a-b)b$ and cancelling yields $a + b = b$. Finally, subtracting b yields $a = 0$ as claimed.

   (b) Proposition: If $a_1 = 3$ and $a_{n+1} = 3a_n + 2$ for all $n \geq 1$, then $a_n = 3^n - 1$ for all n.
   Proof: Induct on n. The base case $n = 1$ is trivial. For the inductive step, suppose $a_n = 3^n - 1$. Then $a_{n+1} = 3a_n + 2 = 3(3^n - 1) + 2 = 3^{n+1} - 1$ as required.

   (c) Proposition: If $a_1 = 2$, and $a_{n+1} = 4a_n - 4a_{n-1}$ for all $n \geq 1$, then $a_n = 2^n$ for all n.
   Proof: Use strong induction on n. The base case $n = 1$ follows since $a_1 = 2 = 2^1$. For the inductive step, suppose $a_k = 2^k$ for all $k \leq n$. Then $a_{n+1} = 4a_n - 4a_{n-1} = 4 \cdot 2^n - 4 \cdot 2^{n-1} = 4 \cdot 2^n - 2 \cdot 2^n = 2 \cdot 2^n = 2^{n+1}$ as required.

   (d) Proposition: All horses are the same color.
   Proof: Induct on n, the number of horses. The base case $n = 1$ is trivial because any 1 horse is the same color as itself. For the inductive step, suppose that any $n - 1$ horses are the same color, and consider any set of n horses. Then the first $n - 1$ horses are the same color, and also the last $n - 1$ horses are the same color, so all of the horses are the same color as the middle $n - 2$ horses. This means all n horses are the same color, as claimed.

   (e) Proposition: All horses are the same color.
   Proof: Induct on n, the number of horses. The base case $n = 1$ is trivial because any 1 horse is the same color as itself. For the inductive step, suppose that any $n + 1$ horses are the same color. Ignoring the last horse yields means that we need to show that n horses are the same color, which is true by the induction hypothesis. Therefore the result holds by induction.

   (f) Proposition: For every positive integer n, $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$.
   Proof: Induct on n. The base case $n = 1$ follows because $1 = \frac{1}{2}(1)(2)$. For the inductive step, suppose that $1+2+3+\cdots+n+(n+1) = \frac{1}{2}(n+1)(n+2)$. Subtracting $n+1$ from both sides yields $1+2+3+\cdots+n = \frac{1}{2}(n+1)(n+2) - (n+1) = \frac{1}{2}n(n+1)$ which is true by the induction hypothesis. Therefore the result holds by induction.

   (g) Proposition: For every positive integer n, $\underbrace{\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots}_{n \text{ terms}} = \frac{3}{2} - \frac{1}{n}$.

   Proof: Induct on n. The base case $n = 1$ follows because $\frac{1}{1 \cdot 2} = \frac{3}{2} - \frac{1}{1}$. For the inductive step, suppose that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} = \frac{3}{2} - \frac{1}{n}$. Then $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} + \frac{1}{n \cdot (n+1)} = \frac{3}{2} - \frac{1}{n} + \frac{1}{n \cdot (n+1)} = \frac{3}{2} - \frac{1}{n+1}$, as claimed.

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2. For each pair of integers (a, b), use the Euclidean algorithm to calculate their greatest common divisor $d = \gcd(a, b)$ and also to find integers x and y such that $d = ax + by$.

   (a) $a = 44$, $b = 12$.

   (b) $a = 461$, $b = 23$.

   (c) $a = 23409$, $b = 2023$.

   (d) $a = 12445$, $b = 5567$.

   (e) $a = 233$, $b = 144$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Prove the following basic properties of divisibility (note that some of these properties are referred to, but not proven, in the course notes; you are expected to give the details of the proof!):

   (a) If a, b are integers, show that $a|b$ if and only if $a|(-b)$.

   (b) If a, b, c are integers such that $a|b$ and $b|c$, show that $a|c$.

   (c) If a, b, m are integers with $m \neq 0$, show that $a|b$ if and only if $(ma)|(mb)$.

   (d) If a, b, c, x, y are integers such that $a|b$ and $a|c$, show that $a|(xb + yc)$.

   (e) If a, b are integers, show that a, b and $a, a + b$ have the same set of common divisors. Deduce that $\gcd(a, b) = \gcd(a, a + b)$.

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4. The Fibonacci-Virahanka numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci-Virahanka sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

   (a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n. [Hint: Use induction.]

   (b) Prove that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ for every positive integer n.

   (c) Prove that $F_{n+1}^2 - F_n F_{n+2} = (-1)^n$ for every positive integer n.

   (d) Prove that $F_{2n+1} = F_{n+1}^2 + F_n^2$ and $F_{2n+2} = F_{n+1}(F_{n+2} + F_n)$ for all $n \geq 1$. [Hint: Show both together by induction.]

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5. Recall that the factorial of n is defined as $n! = n \cdot (n-1) \cdot \cdots \cdot 1$, so for example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. (Note that $0!$ is defined to be 1.)

   (a) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ for every positive integer n.

   (b) Prove that $n! + 1$ and $(n + 1)! + 1$ are relatively prime for every positive integer n. [Hint: Subtract $(n + 1)! + 1$ from a multiple of $n! + 1$.]

   (c) If $n \geq 3$, prove that the integers $n! + 2$, $n! + 3$, ... , $n! + n$ are all composite. Deduce that there are arbitrarily large "prime gaps" (i.e., differences between consecutive prime numbers).
