
Directions: Read ALL of the following directions.

This is an open-notes, open-homework, open-textbook exam. There is no official time limit, but it is suggested that you should be able to solve most of the problems within approximately 5 hours.

There are 11 problems on this exam, with points as indicated.

Justify any answers you give, including computations. You may freely refer to results from in class, from the course notes, course assignments and solutions, and the course textbook, but please make it clear what results you are using.

Proofs and explanations are expected to be clear, concise, and correct.

In problems with multiple parts, you MAY use the results of previous parts in later parts, even if you were unable to solve the earlier parts correctly.

You MAY use a computer for typesetting and to access any material on the course webpage (e.g., the course notes and homework solutions), and to perform computations. Any such computations must be clearly identified and justified as correct.

You MAY NOT use a computer to access any other information.

You MAY ask the instructor for help on any part of the exam, during office hours or via email, and in fact YOU ARE ENCOURAGED to do so.

You MAY NOT discuss the material on this exam with anyone except for the instructor (until after the due date). This includes asking others for hints or solutions, searching for information about the problems online, or posting about the problems on discussion forums.

Please include AND SIGN the following statement with your exam:

I certify that I have neither given nor received any assistance on this exam and have used no resources other than those allowed.

Exams submitted without this certification WILL NOT BE GRADED.

We share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury.

Irving Kaplansky

There are lots of things that are obvious that are not true.

Jim Wiseman

In real life, I assure you, there is no such thing as algebra.

Fran Lebowitz

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1. (10) Identify each of the following statements as true or false (no justification required):

   (a) If V is a vector space that has a linearly independent subset of 3 vectors that does not span V, then $\dim(V) > 3$.

   (b) The functions 1, $\sin x$, $\cos x$, $\sin^2 x$, $\sin x \cos x$, $\cos^2 x$ are linearly dependent on the interval $[0, 1]$.

   (c) If a 7×3 matrix has linearly independent columns, then its reduced row-echelon form has 4 rows of all zeroes.

   (d) There exists a linear transformation $T : \mathbb{R}^4 \to M_{2\times 2}(\mathbb{R})$ whose nullity is 3 and whose rank is 1.

   (e) There exists a vector space V and a linear transformation $T : V \to V$ that is onto but not an isomorphism.

   (f) If $T : V \to W$ is one-to-one and V, W are finite-dimensional, then $\dim(\operatorname{im} T) = \dim(V)$.

   (g) If V and W are finite-dimensional vector spaces, then $\mathcal{L}(V, W)$ is isomorphic to $\mathcal{L}(W, V)$.

   (h) If α and β are ordered bases of the finite-dimensional vector space V and $T : V \to V$ is an isomorphism, then $[T]_\beta^\beta = Q[T]_\alpha^\alpha Q^{-1}$ where $Q = [T]_\alpha^\beta$.

   (i) If A is an $n \times n$ matrix and $\mathbf{x}, \mathbf{c}$ are $n \times 1$ column vectors, then $A\mathbf{x} = \mathbf{c}$ has a unique solution for $\mathbf{x}$ if and only if the nullspace of A consists only of the zero vector.

   (j) If V has ordered bases α and β and $I : V \to V$ is the identity transformation, then $[I]_\alpha^\beta$ is the identity matrix if and only if $\alpha = \beta$.

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2. (20) Calculate the following things (no justification required):

   (a) A basis for the rowspace, column space, and nullspace of $\begin{bmatrix} 2 & 4 & 2 & 0 & 4 & 2 \\ 3 & 6 & 1 & 0 & 2 & 9 \\ 4 & 8 & 3 & 0 & 6 & 7 \end{bmatrix}$.

   (b) A basis for the vector space of 2×3 matrices A such that $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   (c) An extension of the linearly independent set $\{(1, 1, 0, 0), (1, 1, 2, 2)\}$ to a basis of $\mathbb{R}^4$.

   (d) A subset of the vectors $\{(1, 2, 1), (2, 1, 5), (1, -1, 4), (0, -1, 1)\}$ that give a basis of their span.

   (e) A basis for the kernel and image of the linear transformation $S : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ with $S(A) = A + A^T$.

   (f) The matrix $[S]_\beta^\beta$ associated to the linear transformation $S : M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ with $S(A) = A + A^T$, where $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

   (g) The matrix $[T]_\beta^\gamma$ associated to the linear transformation $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ with $P(p) = xp'(x) - p(x)$, where $\beta = \{1 - x, 1 - x^2, 1 - x^3, x^2 + x^3\}$ and $\gamma = \{1, x, x^2, x^3\}$.

   (h) The "angle" between the functions $f(x) = 1 + \sin x$ and $g(x) = 1 - \sin x$ in $C[0, 2\pi]$ with inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\, dx$.

   (i) The coordinate vector $[\mathbf{v}]_\beta$ where $\mathbf{v} = (9, 7, -8)$ and $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the orthogonal basis $\{(-1, 1, 2), (2, 0, 1), (1, 5, -2)\}$ of $\mathbb{R}^3$.

   (j) An orthogonal basis for $P_2(\mathbb{R})$ under the inner product $\langle p, q \rangle = \int_0^6 p(x)q(x)\, dx$.

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3. (5) If W_1 and W_2 are subspaces of V, show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

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4. (5) Suppose $T : V \to W$ is linear. If W_1 is a subspace of W, we define the inverse image $T^{-1}(W_1) = \{\mathbf{v} \in V : T(\mathbf{v}) \in W_1\}$ to be all vectors in V whose image under T lies in W_1. Show that $T^{-1}(W_1)$ is a subspace of V. (Note that T^{-1} is not necessarily a function.)

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5. (5) Suppose $T : V \to W$ is a linear transformation where $\dim(V) = 2022$ and $\dim(W) = 1947$. Prove that $\dim(\ker T) > 70$.

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6. (5) Suppose $T : V \to V$ is a linear transformation such that T^3 is the identity transformation. Prove that T is one-to-one and onto.

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7. (10) Suppose V and W are finite-dimensional vector spaces and $T : V \to W$ is linear.
   (a) If β and γ are ordered bases of V and W respectively such that $[T]_\beta^\gamma$ is the identity matrix, show that T is an isomorphism.
   (b) If T is an isomorphism, show that there exist ordered bases β and γ of V and W respectively such that $[T]_\beta^\gamma$ is the identity matrix.
   (c) If V has ordered basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, prove that $\gamma = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a basis of W if and only if T is an isomorphism.

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8. (10) Suppose that V is a finite-dimensional vector space and $T : V \to V$ is linear.
   (a) Suppose there exists a basis β of V such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. Show that T is a projection map (i.e., that $T^2 = T$).
   (b) Conversely, suppose that T is a projection map. Show that there exists a basis β of V such that $[T]_\beta^\beta$ is a diagonal matrix whose diagonal entries are all 1s and 0s. [Hint: As shown on homework 4, $V = \ker(T) \oplus \operatorname{im}(T)$; take β be a basis of $\ker(T)$ followed by a basis of $\operatorname{im}(T)$.]

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9. (10) Suppose $T : V \to W$ is a linear transformation and $\langle \cdot, \cdot \rangle_W$ is an inner product on W.
   (a) If T is one-to-one, show that $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_V = \langle T(\mathbf{v}_1), T(\mathbf{v}_2) \rangle_W$ is an inner product on V.
   (b) Show that if A is an invertible $n \times n$ real matrix, then the pairing $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T (A^T A) \mathbf{w}$ is an inner product on $\mathbb{R}^n$. [Hint: $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$.]
   (c) Is the map defined in part (a) necessarily an inner product if the assumption that T is one-to-one is dropped? Explain why or why not.

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10. (10) Let $V = \mathbb{R}^2$.
   (a) Show that the pairing $\langle (a, b), (c, d) \rangle = ac + ad + bc + 3bd$ is an inner product on V.
   (b) Show that $(ac + ad + bc + 3bd)^2 \le (a^2 + 2ab + 3b^2)(c^2 + 2cd + 3d^2)$ for any reals a, b, c, d.
   (c) Show that $\sqrt{(a + c)^2 + 2(a + c)(b + d) + 3(b + d)^2} \le \sqrt{a^2 + 2ab + 3b^2} + \sqrt{c^2 + 2cd + 3d^2}$ for any reals a, b, c, d.

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11. (10) Let V be a finite-dimensional inner product space with orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$.
   (a) For any $\mathbf{x} \in V$ show that $\mathbf{x} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{x}, \mathbf{e}_2 \rangle \mathbf{e}_2 + \cdots + \langle \mathbf{x}, \mathbf{e}_n \rangle \mathbf{e}_n$.
   (b) For any $\mathbf{x}, \mathbf{y} \in V$ show that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{e}_1 \rangle \overline{\langle \mathbf{y}, \mathbf{e}_1 \rangle} + \langle \mathbf{x}, \mathbf{e}_2 \rangle \overline{\langle \mathbf{y}, \mathbf{e}_2 \rangle} + \cdots + \langle \mathbf{x}, \mathbf{e}_n \rangle \overline{\langle \mathbf{y}, \mathbf{e}_n \rangle}$.
   (c) For any $\mathbf{x} \in V$ show that $||\mathbf{x}||^2 = |\langle \mathbf{x}, \mathbf{e}_1 \rangle|^2 + |\langle \mathbf{x}, \mathbf{e}_2 \rangle|^2 + \cdots + |\langle \mathbf{x}, \mathbf{e}_n \rangle|^2 = \sum_{i=1}^n |\langle \mathbf{x}, \mathbf{e}_i \rangle|^2$. What theorem from classical geometry does this generalize?
