E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 9, due Mon Apr 11th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a finite-dimensional vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:
   (a) For any scalar λ, the λ-eigenspace of T is a subspace of the generalized λ-eigenspace of T.
   (b) For any λ, a chain of generalized λ-eigenvectors is linearly independent.
   (c) There always exists a basis β of V consisting of generalized eigenvectors of T.
   (d) If all eigenvalues of T lie in F, then there exists a basis β of V of generalized eigenvectors for T.
   (e) There always exists some basis β of V such that the matrix $[T]_\beta^\beta$ is in Jordan canonical form.
   (f) Every matrix $A \in M_{n\times n}(\mathbb{C})$ has a Jordan canonical form.
   (g) If a matrix is diagonalizable, then its Jordan canonical form is diagonal.
   (h) If the Jordan canonical form of a matrix is diagonal, then the matrix is diagonalizable.
   (i) Two matrices are similar if and only if they have equivalent Jordan canonical forms.
   (j) Two matrices are similar if and only if they have the same characteristic polynomials.
   (k) The matrix $\begin{bmatrix} 3 & 2 & 5 \\ 2 & 0 & e \\ 5 & e & \pi \end{bmatrix}$ is diagonalizable.
   (l) The matrix $\begin{bmatrix} 7 & 4-i \\ 4+i & 8 \end{bmatrix}$ is diagonalizable.

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2. Solve the following problems:
   (a) Find a formula for the nth power of the matrix $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$.
   (b) In Diagonalizistan there are two cities: City A and City B. Each year, 2/5 of the residents of City A move to City B, and 2/3 of the residents of City B move to City A; the remaining residents stay in their current city. If in year 0 the populations of Cities A and B are 2000 and 6000 residents respectively, find the populations of the two cities in year n and determine what happens as $n \to \infty$.

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3. Suppose the characteristic polynomial of the 5×5 matrix A is $p(t) = t^3(t-1)^2$.
   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.
   (b) List all possible Jordan canonical forms of A up to equivalence.
   (c) If $\ker(A)$ and $\ker(A - I)$ are both 2-dimensional, what is the Jordan canonical form of A?

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4. Find the Jordan canonical form of each matrix A over $\mathbb{C}$.

   (a) $A = \begin{bmatrix} 5 & 1 \\ -2 & 7 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} -3 & 3 & 1 \\ -7 & 6 & 1 \\ 1 & -1 & 3 \end{bmatrix}$.

   (e) $A = \begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}$.

   (f) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose A is an invertible $n \times n$ matrix and that $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is its characteristic polynomial. Note that $a_0 = (-1)^n \det(A)$ is nonzero.

   (a) If $B = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n)$, show that $AB = I_n$. [Hint: Cayley-Hamilton.]

   (b) Show that there exists a polynomial $q(x)$ of degree at most $n-1$ such that $A^{-1} = q(A)$.

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6. Let $A \in M_{n \times n}(\mathbb{C})$.

   (a) Show that any Jordan-block matrix is similar to its transpose. [Hint: Reverse the Jordan basis.]

   (b) If J is a matrix in Jordan canonical form, show that J is similar to its transpose.

   (c) Show that A is similar to its transpose.

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7. The goal of this problem is to give two proofs of Binet's formula for the Fibonacci-Virahanka numbers defined by the recurrence $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$; the next few terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Explicitly, for $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$, Binet's formula says that $F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$.

   (a) Show that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ and deduce that $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$.

   (b) Find a formula for the nth power of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and use the result to deduce Binet's formula.

   (c) Let W be the space of all real sequences $\{a_n\}_{n \geq 0}$ such that $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$. Show that W is a 2-dimensional vector space over $\mathbb{R}$.

   (d) With notation as in (c), show that the sequences $\{\varphi^n\}_{n \geq 0}$ and $\{\overline{\varphi}^n\}_{n \geq 0}$ are a basis for W. Deduce that there exist constants C and D such that $F_n = C\varphi^n + D\overline{\varphi}^n$ and then deduce Binet's formula.

   Remark: Both of these methods extend generally to solve general linear recurrences of the form $a_{n+1} = C_1 a_n + C_2 a_{n-2} + \cdots + C_k a_{n-k}$ for constants $C_1, \ldots, C_k$. Additionally, the matrix formula in (a) is a good source of other Fibonacci identities.

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8. [Optional] The goal of this problem is to characterize when the limit of matrix powers $\lim_{n\to\infty} A^n$ converges. Suppose J is an $d \times d$ Jordan block matrix with eigenvalue $\lambda \in \mathbb{C}$ and let $N = J - \lambda I_d$ be the matrix with 1s directly above the diagonal and 0s elsewhere.

   (a) Show that $J^n = \lambda^n I_d + \binom{n}{1}\lambda^{n-1}N + \binom{n}{2}\lambda^{n-2}N^2 + \cdots + \binom{n}{d}\lambda^{n-d}N^d$ for each $n \geq 1$.

   (b) Show that $\lim_{n\to\infty} J^n$ exists if and only if $|\lambda| < 1$ or if $\lambda = 1$ and $d = 1$.

   (c) Let A be a square complex matrix. Show that $\lim_{n\to\infty} A^n$ exists if and only if 1 is the only eigenvalue of A of absolute value ≥ 1 and the dimension of the 1-eigenspace equals its multiplicity as a root of the characteristic polynomial.

   (d) Suppose M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1) such that some power of M has all positive entries. Show that $\lim_{n\to\infty} M^n$ converges to a matrix whose columns are all 1-eigenvectors of M. [Hint: Use the results of the challenge problem from homework 8 applied to an appropriate power of M.]
