E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 8, due Mon Apr 4th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $T : V \to V$ be linear. Identify each of the following statements as true or false:

   (a) For any $\lambda \in F$, the set of vectors $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ is a subspace of V.
   (b) If $T(\mathbf{v}) = \lambda\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of T.
   (c) Every linear transformation on V has at least one eigenvector.
   (d) If V is finite-dimensional, every linear transformation on V has at least one eigenvector.
   (e) Any two eigenvectors of T are linearly independent.
   (f) The sum of two eigenvectors of T is also an eigenvector of T.
   (g) The sum of two eigenvalues of T is also an eigenvalue of T.
   (h) If two matrices are similar, then they have the same eigenvectors.
   (i) If two matrices have the same eigenvalues, then they are similar.
   (j) If two matrices are similar, then they have the same eigenvalues.
   (k) If $\dim(V) = n$, then T has at most n distinct eigenvalues in F.
   (l) If $\dim(V) = n$, then T has exactly n distinct eigenvalues in F.
   (m) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the 1-eigenspace of A has dimension 2.
   (n) If the characteristic polynomial of A is $p(t) = t(t-1)^2$, then the only vector $\mathbf{v}$ with $A\mathbf{v} = 3\mathbf{v}$ is $\mathbf{v} = \mathbf{0}$.
   (o) V has a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of T if and only if T is diagonalizable.
   (p) If $\dim(V) = n$ and T has n distinct eigenvalues in F, then T is diagonalizable.
   (q) If $\dim(V) = n$ and T is diagonalizable, then T has n distinct eigenvalues in F.
   (r) If A is a diagonalizable $n \times n$ matrix, then so is $A + I_n$.

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2. For each matrix A over each field F, (i) find all eigenvalues of A over F, (ii) find a basis for each eigenspace of A, and (iii) determine whether or not A is diagonalizable over F and if so find an invertible matrix Q and diagonal matrix D such that $D = Q^{-1}AQ$.

   (a) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{R}$.
   (b) The matrix $\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}$ over $\mathbb{C}$.
   (c) The matrix $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ over $\mathbb{Q}$.
   (d) The matrix $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$ over $\mathbb{C}$.
   (e) The matrix $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$ over $\mathbb{R}$.
   (f) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ over $\mathbb{C}$.

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3. For each operator $T : V \to V$ on each vector space V, (i) find all its eigenvalues and a basis for each eigenspace, and (ii) determine whether the operator is diagonalizable and if so, find a basis for which $[T]_\beta^\beta$ is diagonal:

   (a) The map $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $T(x, y) = (x + 4y, 3x + 5y)$.
   (b) The derivative operator $D : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $D(p) = p'$.
   (c) The transpose map $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ given by $T(M) = M^T$.

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4. Let F be a field and let L and R be the left shift and right shift operators on infinite sequences of elements of F, defined by $L(a_1, a_2, a_3, a_4, \dots) = (a_2, a_3, a_4, \dots)$ and $R(a_1, a_2, a_3, a_4, \dots) = (0, a_1, a_2, a_3, \dots)$.

   (a) Find all of the eigenvalues and a basis for each eigenspace of L.
   (b) Find all of the eigenvalues and a basis for each eigenspace of R.

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5. Suppose A is a matrix with characteristic polynomial $p(t) = (t - 1)^2(t - 2)^4$.

   (a) What are the dimensions of A?
   (b) Find the eigenvalues of A and the possible dimensions of the corresponding eigenspaces.
   (c) Find the determinant and the trace of A.
   (d) If A is diagonalizable, find a diagonal matrix to which A is similar.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

6. The goal of this problem is to find the eigenvalues and eigenvectors of the $n \times n$ "all 1s" matrix over an arbitrary field F. So let $n \geq 2$ and let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$.

   (a) Show that the 0-eigenspace of A has dimension $n - 1$ and find a basis for it.
   (b) If the characteristic of F does not divide n, find the remaining nonzero eigenvalue of A and a basis for the corresponding eigenspace, and show that A is diagonalizable. [Hint: Calculate the trace of A.]
   (c) If the characteristic of F does divide n, show that A is not diagonalizable. [Hint: Note that $\text{char}(F)$ dividing n is the same as saying that $n = 0$ in F.]

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7. Suppose V is a vector space and $S, T : V \to V$ are linear operators on V.

   (a) If S and T commute (i.e., $ST = TS$), show that S maps each eigenspace of T into itself.
   (b) If $\mathbf{v}$ is an eigenvector of T, show that it is also an eigenvector of T^n for any positive integer n.

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8. Suppose V is finite-dimensional and $T : V \to V$ is a projection, so that $T^2 = T$.

   (a) Show that the only possible eigenvalues of T are 0 and 1.
   (b) Show that T is diagonalizable. [Hint: This was done on the midterm exam.]
   (c) Show up to similarity, a projection map on V is uniquely characterized by its rank.

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9. If a linear transformation T on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, prove that T is a scalar multiple of the identity. [Hint: Express one eigenvector in terms of the others, and then apply T.]

   - Remark: This was problem A6 from the 1988 Putnam exam.

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10. [Optional] The goal of this problem is to prove various results about eigenvalues of complex matrices, positive matrices, and stochastic matrices. Let $A \in M_{n\times n}(\mathbb{C})$, define $\rho_i(A)$ to be the sum of the absolute values of the entries in the ith row of A, and define $\rho(A) = \max_{1\le i\le n} \rho_i(A)$.

   (a) Define the ith Gershgorin disk C_i to be the disc in $\mathbb{C}$ centered at $a_{i,i}$ with radius $r_i(A) = \rho_i(A) - |a_{i,i}|$. Prove Gershgorin's disc theorem: every eigenvalue of A is contained in one of the Gershgorin disks of A. [Hint: If $\mathbf{v} = (x_1, \dots, x_n)$ is an eigenvalue where x_k has the largest absolute value among the entries of $\mathbf{v}$, show that $|\lambda x_k - a_{k,k}x_k| \le r_i(A)\,|x_k|$ by noting that λx_k is the kth component of $A\mathbf{v}$.]

   (b) For any eigenvalue λ of $A \in M_{n\times n}(\mathbb{C})$, prove that $|\lambda| \le \rho(A)$.

   (c) Prove that if $A \in M_{n\times n}(\mathbb{R})$ has positive entries and there exists an eigenvalue λ such that $|\lambda| = \rho(A)$, then $\lambda = \rho(A)$ and the λ-eigenspace is 1-dimensional and spanned by the vector $\mathbf{v} = (1, 1, \dots, 1)$. [Hint: Analyze when equality can hold in (a) and (b).]

   (d) If M is a stochastic matrix (i.e., with nonnegative real entries and columns summing to 1), show that every eigenvalue λ of M has $|\lambda| \le 1$. Also show that if M has all entries positive, then the only eigenvalue of M of absolute value 1 is $\lambda = 1$, and the 1-eigenspace has dimension 1. [Hint: Consider M^T.]
