E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 7, due Mon Mar 28th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V with scalar field F. Identify each of the following statements as true or false:

   (a) If V is finite-dimensional and W is any subspace of V, then $\dim(W) = \dim(W^{\perp})$.

   (b) If V has an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, then $||\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4|| = 2$.

   (c) If V has an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $W = \text{span}(\mathbf{e}_1 + 2\mathbf{e}_3)$, then $W^{\perp} = \text{span}(\mathbf{e}_2, 2\mathbf{e}_1 - \mathbf{e}_3)$.

   (d) If $\mathbf{w}^{\perp}$ is a vector in $W^{\perp}$, then the orthogonal projection of $\mathbf{w}^{\perp}$ onto W is $\mathbf{w}^{\perp}$ itself.

   (e) If $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V, then $\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{w}, \mathbf{v}_n \rangle \mathbf{v}_n$ for any $\mathbf{w} \in W$.

   (f) If $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is an orthonormal basis of W, then $\mathbf{w} = \langle \mathbf{v}, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle \mathbf{v}, \mathbf{w}_n \rangle \mathbf{w}_n$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.

   (g) If $A\mathbf{x} = \mathbf{c}$ is an inconsistent system of linear equations, then the best approximation of a solution is given by the solutions $\hat{\mathbf{x}}$ of $A^*\hat{\mathbf{x}} = A^*A\mathbf{c}$.

   (h) If V is finite-dimensional, $\mathbf{v} \in V$, and W is any subspace of V, the vector $\mathbf{w} \in W$ minimizing $||\mathbf{v} - \mathbf{w}||$ is the orthogonal projection of $\mathbf{v}$ into $\mathbf{w}$.

   (i) If $T : V \to V$ is linear, then the adjoint of T exists and is unique.

   (j) If $T : V \to V$ is linear and V is finite-dimensional, then the adjoint of T exists and is unique.

   (k) If $T : V \to F$ is linear and V is finite-dimensional, then there exists $\mathbf{w} \in V$ such that $T(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v} \in V$.

   (l) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(S + 2T)^* = S^* + 2T^*$.

   (m) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(S + iT)^* = S^* + iT^*$.

   (n) For any $S, T : V \to V$ such that S^* and T^* exist, we have $(ST)^* = S^*T^*$.

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2. Calculate the following things (assume any unspecified inner product is the standard one):

   (a) Write $\mathbf{v} = (-5, 5, -6)$ as a linear combination of the orthogonal basis $(i, -i, 0)$, $(1, 1, 2i)$, $(i, i, 1)$ of $\mathbb{C}^3$.

   (b) A basis for $W^{\perp}$, if $W = \text{span}[(1, 1, 1, 1), (2, 3, 4, 1)]$ inside $\mathbb{R}^4$.

   (c) A basis for $W^{\perp}$, if $W = \text{span}[(1, 1, 2i), (1, -i, 4)]$ inside $\mathbb{C}^3$. [Hint: Over $\mathbb{C}$, compute the complex conjugate of the nullspace.]

   (d) The orthogonal decomposition $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ of $\mathbf{v} = (2, 0, 11)$ into $W = \text{span}[\frac{1}{3}(1, 2, 2), \frac{1}{3}(2, -2, 1)]$ inside $\mathbb{R}^3$. Also, verify the relation $||\mathbf{v}||^2 = ||\mathbf{w}||^2 + ||\mathbf{w}^{\perp}||^2$.

   (e) An orthogonal basis for $W = \text{span}[x, x^2, x^3]$, and the orthogonal projection of $\mathbf{v} = 1 + 2x^2$ into W, with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx$.

   (f) The quadratic polynomial $p(x) \in P_2(\mathbb{R})$ that minimizes the expression $\int_0^1 [p(x) - \sqrt{x}]^2\, dx$.

   (g) The least-squares solution to the inconsistent system $x + 3y = 9$, $3x + y = 5$, $x + y = 2$.

   (h) The least-squares line $y = mx + b$ approximating the points $\{(4, 7), (11, 21), (15, 29), (19, 35), (30, 49)\}$. (Give three decimal places.)

   (i) The least-squares quadratic $y = ax^2 + bx + c$ approximating the points $\{(-2, 22), (-1, 11), (0, 4), (1, 3), (2, 13)\}$. (Give three decimal places.)

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

3. Let V be an inner product space with scalar field F. The goal of this problem is to prove the so-called "polarization identities".

   (a) If $F = \mathbb{R}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} ||\mathbf{v} + \mathbf{w}||^2 - \frac{1}{4} ||\mathbf{v} - \mathbf{w}||^2$.

   (b) If $F = \mathbb{C}$, prove that $\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \left|\left|\mathbf{v} + i^k \mathbf{w}\right|\right|^2$.

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4. Let V be a finite-dimensional inner product space and W be a subspace of V.

   (a) Prove that $W \cap W^{\perp} = \{\mathbf{0}\}$ and deduce that $V = W \oplus W^{\perp}$. [Hint: Use $\dim(W) + \dim(W^{\perp}) = \dim(V)$.]

   (b) Let $T : V \to W$ be the function defined by setting $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ for $\mathbf{w} \in W$ and $\mathbf{w}^{\perp} \in W^{\perp}$. Prove that T is linear, that $T^2 = T$, that $\mathrm{im}(T) = W$, and that $\ker(T) = W^{\perp}$. Conclude that T is projection onto the subspace W with kernel $W^{\perp}$.

   (c) Show that $(W^{\perp})^{\perp} = W$. [Hint: For $(W^{\perp})^{\perp} \subseteq W$, write $\mathbf{v} \in (W^{\perp})^{\perp}$ as $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ and find $\langle \mathbf{v}, \mathbf{w}^{\perp} \rangle$.]

   - <u>Remark</u>: Most of these results break down if V is infinite-dimensional (see problem 8 for examples).

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5. Suppose V is an inner product space (not necessarily finite-dimensional) and $T : V \to V$ is a linear transformation possessing an adjoint T^*. We say T is <u>Hermitian</u> (or <u>self-adjoint</u>) if $T = T^*$, and that T is <u>skew-Hermitian</u> if $T = -T^*$.

   (a) Show that T is Hermitian if and only if iT is skew-Hermitian.

   (b) Show that $T + T^*$, T^*T, and TT^* are all Hermitian, while $T - T^*$ is skew-Hermitian.

   (c) Show that T can be written as $T = S_1 + iS_2$ for unique Hermitian transformations S_1 and S_2.

   (d) Suppose T is Hermitian. Prove that $\langle T(\mathbf{v}), \mathbf{v} \rangle$ is a real number for any vector $\mathbf{v}$.

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6. Suppose V is an inner product space over the field F (where $F = \mathbb{R}$ or $\mathbb{C}$) and $T : V \to V$ is linear. We say T is a "distance-preserving map" on V if $||T\mathbf{v}|| = ||\mathbf{v}||$ for all $\mathbf{v}$ in V, and we say T is an "angle-preserving map" on V if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle T\mathbf{v}, T\mathbf{w} \rangle$ for all $\mathbf{v}$ and $\mathbf{w}$ in V.

   (a) Prove that T is distance-preserving if and only if it is angle-preserving. [Hint: Use problem 3.]

   A map $T : V \to V$ satisfying the distance- and angle-preserving conditions is called a (linear) <u>isometry</u>.

   (b) Show that the transformations $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $S(x, y, z) = (z, -x, y)$ and $T(x, y, z) = \frac{1}{3}(x + 2y + 2z, 2x + y - 2z, 2x - 2y + z)$ are both isometries under the usual dot product.

   (c) Show that isometries are one-to-one.

   (d) Show that isometries preserve orthogonal and orthonormal sets.

   (e) Suppose T^* exists. Prove that T is an isometry if and only if T^*T is the identity transformation.

   (f) We say that a matrix $A \in M_{n \times n}(F)$ is <u>unitary</u> if $A^{-1} = A^*$. Show that the isometries of F^n (with its usual inner product) are precisely those maps given by left-multiplication by a unitary matrix.

   - <u>Remark</u>: Notice that $A \in M_{n \times n}(\mathbb{C})$ is unitary if and only if the columns of A are an orthonormal basis of $\mathbb{C}$. Thus, the result of part (f) can equivalently be thought of as saying that the distance-preserving maps on $\mathbb{C}^n$ (or $\mathbb{R}^n$) are simply changes of basis from one orthonormal basis (the columns of A) to another (the standard basis).

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7. Let $V = C[0,2\pi]$ with inner product $\langle f,g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$. Also define $\varphi_0(x) = \dfrac{1}{\sqrt{2\pi}}$, and for positive integers k set $\varphi_{2k-1}(x) = \dfrac{1}{\sqrt{\pi}}\cos(kx)$ and $\varphi_{2k}(x) = \dfrac{1}{\sqrt{\pi}}\sin(kx)$.

   (a) Show that $\{\varphi_0, \varphi_1, \varphi_2, \dots\}$ is an orthonormal set in V.

   (b) Let $f(x) = x$. Find $||f||$ and $\langle f, \varphi_n\rangle$ for each $n \geq 0$. (You don't need to give details of the integral evaluations, just the resulting values.)

   (c) With $f(x) = x$, assuming that $f(x) = \sum_{k=0}^{\infty} \langle f, \varphi_k\rangle\, \varphi_k(x)$, derive Leibniz's formula $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$. [Hint: Set $x = \pi/2$.]

   (d) With $f(x) = x$, assuming that $||f||^2 = \sum_{k=0}^{\infty} \langle f, \varphi_k\rangle^2$ (see problem 11 of the midterm for why this is a reasonable statement), find the exact value of $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$.

Remarks: The identity $||f||^2 = \sum_{k=0}^{\infty} \langle f, \varphi_k\rangle^2$ is known as Parseval's identity. The problem of computing the value of the infinite sum $\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ is known as the Basel problem. The correct value was (famously) first found by Euler, who evaluated the sum by decomposing the function $\dfrac{\sin(\pi x)}{\pi x}$ as the infinite product $\prod_{n=1}^{\infty}(1 - \dfrac{x^2}{n^2})$ and then comparing the power series coefficients of both sides.

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8. [Challenge] The goal of this problem is to illustrate some complexities with orthogonal complements, projections, and adjoints in infinite-dimensional spaces. Let V be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \dots)$ with only finitely many nonzero terms, with inner product given by $\langle \{a_i\}, \{b_i\}\rangle = \sum_{i=1}^{\infty} a_i b_i$. (Note that this converges since only finitely many terms in the sum are nonzero.) Let $\mathbf{e}_i$ be the ith unit coordinate vector and observe that $\{\mathbf{e}_i\}_{i\geq 1}$ is an orthonormal basis for V.

   (a) For each $n \geq 2$, let $\mathbf{v}_n = \mathbf{e}_1 - \mathbf{e}_n$ and define $W = \text{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \dots)$. Show that $\mathbf{e}_1 \notin W$ so that W is a proper subspace of V, but that $W^\perp = \{\mathbf{0}\}$. Deduce that $W^\perp + W \neq V$ and that $(W^\perp)^\perp \neq W$, yielding counterexamples to problems 4(a) and 4(c) in the infinite-dimensional case.

   (b) Let $T : V \to V$ be the linear transformation defined by setting $T(\mathbf{e}_n) = \sum_{i=1}^{n} \mathbf{e}_i$ for each $i \geq 1$. If T had an adjoint $T^* : V \to V$, show that infinitely many components of $T^*(\mathbf{e}_1)$ would be nonzero, and deduce that T^* cannot exist.
