E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 6, due Thu Mar 3rd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Identify each of the following statements as true or false:
   (a) An inner product is linear in each of its components.
   (b) There is exactly one inner product on $\mathbb{R}^n$.
   (c) In any inner product space, $\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$.
   (d) In any inner product space, $||\mathbf{v} + \mathbf{w}|| \geq ||\mathbf{v}|| + ||\mathbf{w}||$.
   (e) In any inner product space, if $\langle \mathbf{v}, 2\mathbf{v} \rangle = 0$ then $\mathbf{v} = \mathbf{0}$.
   (f) The Cauchy-Schwarz inequality holds in every inner product space.
   (g) The triangle inequality holds in real inner product spaces but not complex inner product spaces.
   (h) The zero vector is orthogonal to itself.
   (i) In any inner product space, if $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, 2\mathbf{w} \rangle$ then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.
   (j) If V is a complex vector space, the vectors $\mathbf{v}$ and $i\mathbf{v}$ are always orthogonal.
   (k) $(0, 0, 0, 0)$, $(5, -2, 5, 0)$, $(1, -5, -1, 0)$, $(2, 0, 2, 2)$ is an orthogonal set in $\mathbb{R}^4$, with the standard dot product.
   (l) $\frac{1}{9}(4, -1, 8)$, $\frac{1}{9}(7, -4, -4)$, $\frac{1}{9}(4, 8, 1)$ is an orthonormal basis of $\mathbb{R}^3$, with the standard dot product.
   (m) An orthogonal set of vectors is linearly independent.
   (n) An orthonormal set of vectors is linearly independent.
   (o) Every finite-dimensional inner product space has an orthonormal basis.

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2. For each of the following pairings, determine (with justification) whether or not it is an inner product on the given vector space:
   (a) The pairing $\langle A, B \rangle = \mathrm{tr}(A + B)$ on $M_{2\times 2}(\mathbb{R})$.
   (b) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{R}^2$.
   (c) The pairing $\langle (a, b), (c, d) \rangle = 5ac + 3bc + 3ad + 4bd$ on $\mathbb{C}^2$.
   (d) The pairing $\langle (a, b), (c, d) \rangle = ac$ on $\mathbb{R}^2$.
   (e) The pairing $\langle f, g \rangle = \int_0^1 f'(x)g(x)\, dx$ on $C[0, 1]$.

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3. For each pair of vectors $\mathbf{v}, \mathbf{w}$ in the given inner product space, compute $\langle \mathbf{v}, \mathbf{w} \rangle$, $||\mathbf{v}||$, $||\mathbf{w}||$, and $||\mathbf{v} + \mathbf{w}||$, and verify the Cauchy-Schwarz and triangle inequalities for $\mathbf{v}$ and $\mathbf{w}$:
   (a) $\mathbf{v} = (1, 2, 2, 4)$ and $\mathbf{w} = (4, 1, 4, 4)$ in $\mathbb{R}^4$ with the standard inner product.
   (b) $\mathbf{v} = (i, -i, 1 + i)$ and $\mathbf{w} = (2 - i, 4, -2i)$ in $\mathbb{C}^3$ with the standard inner product.
   (c) $\mathbf{v} = e^t$ and $\mathbf{w} = e^{2t}$ in $C[0, 1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$.

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4. For each list S of vectors in the given inner product space, apply Gram-Schmidt to calculate an orthogonal basis for $\mathrm{span}(S)$:
   (a) $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (1, 2, 0)$, $\mathbf{v}_3 = (1, 2, 3)$ in $\mathbb{R}^3$ under the standard dot product.
   (b) $\mathbf{v}_1 = (2, 4, -4)$, $\mathbf{v}_2 = (1, -1, 4)$, $\mathbf{v}_3 = (1, 1, 1)$ in $\mathbb{R}^3$ under the standard dot product.
   (c) $\mathbf{v}_1 = x$, $\mathbf{v}_2 = x^2$, $\mathbf{v}_3 = x^3$ in $C[-1, 1]$ under the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)\, dx$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Let V be an inner product space.

   (a) If $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are two inner products on V, show that $\langle\cdot,\cdot\rangle_3 = \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$ is also an inner product on V, where $\langle\mathbf{v},\mathbf{w}\rangle_3 = \langle\mathbf{v},\mathbf{w}\rangle_1 + \langle\mathbf{v},\mathbf{w}\rangle_2$.

   (b) If $\langle\cdot,\cdot\rangle_1$ is an inner product on V and c is a positive real number, show that $\langle\cdot,\cdot\rangle_3 = c\langle\cdot,\cdot\rangle_1$ is also an inner product on V, where $\langle\mathbf{v},\mathbf{w}\rangle_3 = c\langle\mathbf{v},\mathbf{w}\rangle_1$.

   (c) Does the collection of inner products on V form a vector space under the natural addition and scalar multiplication described above? Explain why or why not.

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6. Prove the following inequalities:

   (a) Prove that $(a_1 + a_2 + \cdots + a_n) \cdot (\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) \geq n^2$ for any positive real numbers $a_1, a_2, \ldots, a_n$, with equality if and only if all of the a_i are equal.

   (b) If a, b, c, d are real numbers with $a^2 + b^2 + c^2 + d^2 \leq 5$, show that $a + 2b + 3c + 4d \leq 5\sqrt{6}$.

   (c) Prove Nesbitt's inequality: for any positive real numbers a, b, c it is true that $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$. [Hint: Apply Cauchy-Schwarz to $(\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ and $(1/\sqrt{a+b}, 1/\sqrt{b+c}, 1/\sqrt{c+a})$.]

   (d) Prove the following generalization of Cauchy-Schwarz: if $\langle\cdot,\cdot\rangle$ is an inner product on the vector space V, it is true that $\left[\sum_{j=1}^n \langle\mathbf{v}_j,\mathbf{w}_j\rangle\right]^2 \leq \left[\sum_{j=1}^n \langle\mathbf{v}_j,\mathbf{v}_j\rangle\right] \cdot \left[\sum_{j=1}^n \langle\mathbf{w}_j,\mathbf{w}_j\rangle\right]$ for any vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ in V. [Hint: Apply Cauchy-Schwarz to the space $\tilde{V}$ of n-tuples of elements of V.]

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7. Find all continuous positive functions $f(x)$ on the interval $[0,1]$ such that $\int_0^1 f(x)\,dx = 1$, $\int_0^1 xf(x)\,dx = \alpha$, and $\int_0^1 x^2 f(x)\,dx = \alpha^2$ where α is a given real number. [Hint: Consider $\langle g, h\rangle = \int_0^1 g(x)h(x)\cdot f(x)\,dx$.]

   - Remark: This was problem A2 from the 1964 Putnam exam.

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8. [Challenge] The goal of this problem is to give an example of an inner product space that has no orthonormal basis. Let $V = \ell^2(\mathbb{R})$ be the vector space of infinite real sequences $\{a_i\}_{i\geq 1} = (a_1, a_2, \ldots)$ such that $\sum_{i=1}^\infty a_i^2$ is finite, under componentwise addition and scalar multiplication.

   (a) Show that the pairing $\langle\{a_i\}_{i\geq 1}, \{b_i\}_{i\geq 1}\rangle = \sum_{i=1}^\infty a_i b_i$ is an inner product on V. (Make sure to justify why this sum converges.)

   (b) Let $\mathbf{v}_i \in V$ be the sequence with a 1 in the ith component and 0s elsewhere. Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \ldots\}$ is an orthonormal set in V and that the only vector $\mathbf{w}$ orthogonal to all of the $\mathbf{v}_i$ is the zero vector. Deduce that S is a maximal orthonormal set of V that is not a basis of V.

   Part (b) shows that Gram-Schmidt does not necessarily construct an orthonormal basis of V. In fact, V has no orthonormal basis at all.

   (c) Suppose V has an orthonormal basis $\{\mathbf{e}_i\}_{i\in I}$ for some indexing set I (which is necessarily infinite), and choose a countably infinite subset $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \ldots$. Show that the sum $\mathbf{v} = \sum_{k=1}^\infty 2^{-k}\mathbf{e}_k$ is a well-defined vector in V that cannot be written as a (finite) linear combination of the basis $\{\mathbf{e}_i\}_{i\in I}$. [Hint: Show that $||\mathbf{v}||^2 = \lim_{n\to\infty} \left|\left|\sum_{k=1}^n 2^{-k}\mathbf{e}_k\right|\right|^2$ is finite.]

   - Remark: The point here is that because our definition of span and basis only allows us to use finite linear combinations, these definitions are not well suited to handle infinite-dimensional spaces like $\ell^2(\mathbb{R})$. However, it is possible (by exploiting the fact that ℓ^2 is a topologically-complete metric space) to deal with these issues and define a "Schauder basis" that allows the use of infinite sums, which amounts to viewing ℓ^2 as a Hilbert space.
