E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 5, due Thu Feb 24th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that the vector spaces U, V, W are finite-dimensional over the field F, the bases $\alpha, \beta, \gamma, \delta$ are ordered, and that S, T are linear transformations. Identify each of the following statements as true or false:
   (a) If $\dim(V) = m$ and $\dim(W) = n$, then $[T]_\beta^\gamma$ is an element of $M_{m\times n}(F)$.
   (b) If $[S]_\alpha^\beta = [T]_\alpha^\beta$ then $S = T$.
   (c) If $[T]_\alpha^\beta = [T]_\gamma^\delta$ then $\alpha = \gamma$ and $\beta = \delta$.
   (d) If $S : V \to W$ and $T : V \to W$ then $[S + T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$.
   (e) If $T : V \to W$ and $\mathbf{v} \in V$, then $[T]_\alpha^\beta[\mathbf{v}]_\beta = [T\mathbf{v}]_\alpha$.
   (f) If $S : V \to W$ and $T : U \to V$, then $[ST]_\alpha^\gamma = [S]_\beta^\gamma[T]_\alpha^\beta$.
   (g) If $T : V \to V$ has an inverse T^{-1}, then $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$.
   (h) If $T : V \to V$ has an inverse T^{-1}, then for any $\mathbf{v} \in V$, $[T^{-1}\mathbf{v}]_\gamma = ([T]_\gamma^\beta)^{-1}[\mathbf{v}]_\beta$.
   (i) If $T : V \to V$ and $[T]_\beta^\gamma$ is the identity matrix, then T must be the identity transformation.
   (j) If $T : V \to V$ and $[T]_\beta^\gamma$ is the zero matrix, then T must be the zero transformation.
   (k) The space $\mathcal{L}(V, W)$ of all linear transformations from V to W has dimension $\dim V \cdot \dim W$.
   (l) If A is an $m \times n$ matrix of rank r, then the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension r.
   (m) If A is an $m \times n$ matrix and the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $\mathrm{rank}(A) < n$.
   (n) If A is an $n \times n$ matrix of rank n, then the equation $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.
   (o) If the columns of A are all scalar multiples of some vector $\mathbf{v}$, then $\mathrm{rank}(A) \le 1$.
   (p) For any $T : V \to V$, there always exists an invertible matrix Q such that $[T]_\beta^\beta = Q^{-1}[T]_\alpha^\alpha Q$.
   (q) For any $T : V \to V$, if $P = [I]_\beta^\gamma$, then it is true that $[T]_\gamma^\gamma = P[T]_\beta^\beta P^{-1}$.

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2. For each linear transformation T and given bases β and γ, find $[T]_\beta^\gamma$:
   (a) $T : \mathbb{C}^2 \to \mathbb{C}^3$ given by $T(a, b) = \langle a - b, b - 2a, 3b\rangle$, with β and γ the standard bases.
   (b) The trace map from $M_{2\times2}(\mathbb{R}) \to \mathbb{R}$ with $\beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right\}$ and $\gamma = \{1\}$.
   (c) $T : \mathbb{Q}^4 \to P_4(\mathbb{Q})$ given by $T(a, b, c, d) = a + (a + b)x + (a + 3c)x^2 + (2a + d)x^3 + (b + 5c + d)x^4$, with β the standard basis and $\gamma = \{x^3, x^2, x^4, x, 1\}$.
   (d) $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ given by $T(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A$ with $\beta = \gamma$ the standard basis.
   (e) The projection map (see problem 8 of homework 4) on $\mathbb{R}^3$ that maps the vectors $\langle 1, 2, 1\rangle$ and $\langle 0, -3, 1\rangle$ to themselves and sends $\langle 1, 1, 1\rangle$ to the zero vector, with $\beta = \gamma = \{\langle 1, 2, 1\rangle, \langle 0, -3, 1\rangle, \langle 1, 1, 1\rangle\}$.
   (f) The same map as in part (e), but relative to the standard basis for $\mathbb{R}^3$.

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3. Let $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ be given by $T(p) = x^2p''(x)$.
   (a) With the bases $\alpha = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2, x^3, x^4\}$, find $[T]_\alpha^\gamma$.
   (b) If $q(x) = 1 - x^2 + 2x^3$, compute $[q]_\alpha$ and $[T(q)]_\gamma$ and verify that $[T(q)]_\gamma = [T]_\alpha^\gamma[q]_\alpha$.

   Notice that $T = SU$ where $U : P_3(\mathbb{R}) \to P_1(\mathbb{R})$ has $U(p) = p''(x)$ and $S : P_1(\mathbb{R}) \to P_4(\mathbb{R})$ has $S(p) = x^2p(x)$.
   (c) With $\beta = \{1, x\}$, compute the associated matrices $[S]_\beta^\gamma$, and $[U]_\alpha^\beta$ and then verify that $[T]_\alpha^\gamma = [S]_\beta^\gamma[U]_\alpha^\beta$.
   (d) Which of S, T, and U are onto? One-to-one? Isomorphisms?

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4. Suppose $V = P_3(\mathbb{R})$, with standard basis $\beta = \{1, x, x^2, x^3\}$, and let $T : V \to V$ be the linear transformation with $T(1) = 1 - x + x^2 - x^3$, $T(x) = 2x - x^3$, and $T(x^2) = 3 + x - x^3$, and $T(x^3) = 1 - x^2$.

   (a) Find $[T]_\beta^\beta$.

   Now let γ be the ordered basis $\gamma = \{x^3, x^2, x+1, x\}$.

   (b) Find the change-of-basis matrix $Q = [I]_\beta^\gamma$ and its inverse.

   (c) For $\mathbf{v} = 2 - x - 2x^2 + x^3$, compute $[\mathbf{v}]_\beta$, $[\mathbf{v}]_\gamma$, and verify that $[\mathbf{v}]_\gamma = Q[\mathbf{v}]_\beta$.

   (d) Find $[T]_\beta^\gamma$, $[T]_\gamma^\beta$, and $[T]_\gamma^\gamma$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. Suppose that $T : V \to V$ is a linear transformation on a finite-dimensional vector space.

   (a) If β and γ are two ordered bases of V, show that $\det([T]_\beta^\beta) = \det([T]_\gamma^\gamma)$.

   Per part (a), we define $\det(T)$ to be $\det([T]_\beta^\beta)$ for any choice of ordered basis β.

   (b) Show that T is an isomorphism if and only if $\det(T)$ is nonzero.

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6. Let F be a field and $n \geq 2$ be an integer. Recall that we say two matrices A and B are similar if there exists an invertible matrix Q with $B = Q^{-1}AQ$.

   (a) Show that if A and B are similar matrices in $M_{n\times n}(F)$, then $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. [Hint: You may use the fact that $\mathrm{tr}(CD) = \mathrm{tr}(DC)$.]

   (b) Show that "being similar" is an equivalence relation on $M_{n\times n}(F)$.

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7. Let V be a vector space and $T : V \to V$ be linear.

   (a) If V is finite-dimensional and $\ker(T) \cap \mathrm{im}(T) = \{\mathbf{0}\}$, prove in fact that $V = \ker(T) \oplus \mathrm{im}(T)$. [Hint: Use problem 4 from homework 3.]

   (b) Show that the result of (a) is not necessarily true if V is infinite-dimensional.

   (c) If V is finite-dimensional and $V = \ker(T) + \mathrm{im}(T)$, prove in fact that $V = \ker(T) \oplus \mathrm{im}(T)$.

   (d) Show that the result of (c) is not necessarily true if V is infinite-dimensional.

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8. [Challenge] The goal of this problem is to demonstrate an unintuitive construction using infinite bases.

   (a) Show that $\dim_{\mathbb{Q}} \mathbb{R} = \dim_{\mathbb{Q}} \mathbb{C}$. Deduce that there exists a $\mathbb{Q}$-vector space isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$. [Hint: Use the fact that finite-dimensional $\mathbb{Q}$-vector spaces are countable.]

   We will now use this isomorphism $\varphi : \mathbb{C} \to \mathbb{R}$ to define a different vector space structure on $\mathbb{C}$. Intuitively, the idea is to start with the set $\mathbb{R}$ as a vector space over itself, and then use the isomorphism φ^{-1} to relabel the vectors as complex numbers, but keep the scalars as real numbers.

   (b) Let V be the set of complex numbers with the addition operation $z_1 \oplus z_2 = z_1 + z_2$ and scalar multiplication $\alpha \odot z = \varphi^{-1}[\alpha\varphi(z)]$ for $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$. Show that $(V, \oplus, \odot)$ is an $\mathbb{R}$-vector space.

   (c) Using the vector space structure defined in (b), show that $\dim_{\mathbb{R}} V = 1$.

   - Remark: The point of (c) is that by changing the definition of scalar multiplication, we can make $\mathbb{C}$ into a 1-dimensional $\mathbb{R}$-vector space. By doing a similar thing in the reverse order, we could even make $\mathbb{R}$ into a 2-dimensional $\mathbb{C}$-vector space.
