E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 4, due Thu Feb 17th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Assume that V and W are arbitrary vector spaces, not necessarily finite-dimensional, over the scalar field F. Identify each of the following statements as true or false:

   (a) If $T : V \to W$ is a linear transformation, then $T(\mathbf{0}_V) = \mathbf{0}_W$.
   (b) If $T : V \to W$ has $T(\mathbf{a}+\mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b})$ for every $\mathbf{a}, \mathbf{b} \in V$ then T is a linear transformation.
   (c) If $T : V \to W$ has $T(r\mathbf{a}) = rT(\mathbf{a})$ for every $r \in F$ and every $\mathbf{a} \in V$ then T is a linear transformation.
   (d) If T is a linear transformation such that $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$, then T is one-to-one.
   (e) If $T : V \to V$ is a one-to-one linear transformation, then $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.
   (f) If $\dim(\text{im}(T)) = \dim(W)$, then T is onto.
   (g) If $T : V \to W$ is a linear transformation and S is a linearly independent subset of V, then $T(S) = \{T(\mathbf{s}) : \mathbf{s} \in S\}$ is a linearly independent subset of W.
   (h) If $T : V \to W$ is a linear transformation and S is a spanning set for V, then $T(S) = \{T(\mathbf{s}) : \mathbf{s} \in S\}$ is a spanning set for W.
   (i) If $T : V \to W$ is a linear transformation and S is a basis for V, then $T(S) = \{T(\mathbf{s}) : \mathbf{s} \in S\}$ is a basis for W.
   (j) For any $\mathbf{v}_1, \mathbf{v}_2 \in V$ and any $\mathbf{w}_1, \mathbf{w}_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$.
   (k) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 2 and whose rank is 2.
   (l) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 4 and whose rank is 1.
   (m) There exists a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose nullity is 1 and whose rank is 4.
   (n) If $T : V \to W$ is linear and for any $\mathbf{w} \in W$ there is a unique $\mathbf{v} \in V$ with $T(\mathbf{v}) = \mathbf{w}$, then T is an isomorphism.
   (o) If V is isomorphic to W, then $\dim(V) = \dim(W)$.

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2. For each map $T : V \to W$, determine whether or not T is a linear transformation from V to W, and if it is not, identify at least one property that fails:

   (a) $V = W = \mathbb{R}^4$, $T(a, b, c, d) = (a - b, b - c, c - d, d - a)$.
   (b) $V = W = \mathbb{R}^2$, $T(a, b) = (a, b^2)$.
   (c) $V = W = M_{2\times 2}(\mathbb{Q})$, $T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} A - A \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.
   (d) $V = W = \mathbb{C}[x]$, $T(p(x)) = p(x^2) - xp'(x)$.
   (e) $V = W = M_{4\times 4}(\mathbb{F}_2)$, $T(A) = Q^{-1}AQ$, for a fixed 4×4 matrix Q.
   (f) $V = W = M_{4\times 4}(\mathbb{R})$, $T(A) = A^{-1}QA$, for a fixed 4×4 matrix Q.

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3. For each map $T : V \to W$, (i) show that T is a linear transformation, (ii) find bases for the kernel and image of T, (iii) compute the nullity and rank of T and verify the conclusion of the nullity-rank theorem, and (iv) identify whether T is one-to-one, onto, or an isomorphism.

   (a) $T : \mathbb{Q}^2 \to \mathbb{Q}^3$ defined by $T(a, b) = \langle a + b, 2a + 2b, a + b\rangle$.
   (b) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$.
   (c) $T : P_2(\mathbb{C}) \to P_3(\mathbb{C})$ defined by $T(p) = xp(x) + p'(x)$.
   (d) $T : P_3(\mathbb{F}_3) \to P_4(\mathbb{F}_3)$ defined by $T(p) = x^3 p''(x)$. [Warning: Note that $3 = 0$ in $\mathbb{F}_3$.]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Suppose that $T : V \to W$ is a linear transformation.
   (a) If T is onto, show that $\dim(W) \le \dim(V)$.
   (b) If T is one-to-one, show that T is an isomorphism from V to $\mathrm{im}(T)$, and deduce that $\dim(V) \le \dim(W)$.

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5. Suppose $\dim(V) = n$ and that $T : V \to V$ is a linear transformation with $T^2 = 0$: in other words, that $T(T(\mathbf{v})) = \mathbf{0}$ for every vector $\mathbf{v} \in V$.
   (a) Show that $\mathrm{im}(T)$ is a subspace of $\ker(T)$.
   (b) Show that $\dim(\mathrm{im}(T)) \le n/2$.

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6. Let F be a field. By using the division algorithm for polynomials, it can be shown that any polynomial in $P_d(F)$ having more than d roots must be the zero polynomial. (You may assume that fact for this problem.) Now let $a_0, a_1, \ldots, a_d$ be distinct elements of F and consider the linear transformation $T : P_d(F) \to F^{d+1}$ given by $T(p) = (p(a_0), p(a_1), \ldots, p(a_d))$.
   (a) Show that $\ker(T) = \{\mathbf{0}\}$ and deduce that T is an isomorphism.
   (b) Conclude that, for any list of $d+1$ points $(a_0, b_0), \ldots, (a_d, b_d)$ with distinct first coordinates, there exists a unique polynomial of degree at most d having the property that $p(a_i) = b_i$ for each $0 \le i \le d$.

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7. Let F be a field and let V be the vector space of infinite sequences $\{a_n\}_{n\ge 1} = (a_1, a_2, a_3, a_4, \ldots)$ of elements of F. Define the <u>left-shift operator</u> $L : V \to V$ via $L(a_1, a_2, a_3, a_4, \ldots) = (a_2, a_3, a_4, a_5, \ldots)$ and the <u>right-shift operator</u> $R : V \to V$ via $R(a_1, a_2, a_3, a_4, \ldots) = (0, a_1, a_2, a_3, \ldots)$ and note both L and R are linear.
   (a) Show that L is onto but not one-to-one while R is one-to-one but not onto.
   (b) Deduce that on infinite-dimensional vector spaces, the conditions of being one-to-one, being onto, and being an isomorphism are not in general equivalent.
   (c) Verify that $L \circ R$ is the identity map on V, but that $R \circ L$ is not the identity map on V.
   (d) Deduce that on infinite-dimensional vector spaces, a linear transformation with a left inverse or a right inverse need not have a two-sided inverse.

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8. A linear transformation $T : V \to V$ such that $T^2 = T$ is called a <u>projection map</u>. The goal of this problem is to give some other descriptions of projection maps.
   (a) Suppose that $T : V \to V$ has the property that there exists a subspace W such that $\mathrm{im}(T) = W$ and T is the identity map when restricted to W. Show that T is a projection map (it is called the <u>projection onto the subspace W</u>).
   (b) Conversely, suppose T is a projection map. Show that T is a projection onto the subspace $W = \mathrm{im}(T)$.
   (c) Suppose that T is a projection map. Prove that $V = \ker(T) \oplus \mathrm{im}(T)$. [Hint: Try decomposing a vector $\mathbf{v} = [\mathbf{v} - T(\mathbf{v})] + T(\mathbf{v})$.]
   - <u>Remark</u>: Projection maps are so named because they represent the geometric idea of projection. For example, in the event that $W = \mathrm{im}(T)$ is one-dimensional, the corresponding projection map T represents projecting onto that line.

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9. [Challenge] Let V be an n-dimensional vector space over F, let $T : V \to V$ be linear, and suppose k is a fixed integer with $1 \le k < n$. If $T(W) \subseteq W$ for all subspaces W with $\dim_F W = k$, prove that T is multiplication by some scalar. [Hint: Reduce to the case $k = 1$.]
