E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 11, due Wed Apr 27th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Identify each of the following statements as true or false, where V is a finite-dimensional inner product space:
   (a) The function $Q(x, y) = xy$ on $\mathbb{R}^2$ is a quadratic form.
   (b) The function $Q(x, y) = x^2 - 4xy + xyz + z^2$ on $\mathbb{R}^3$ is a quadratic form.
   (c) The function $Q(f) = \int_0^1 x\, f(x)^2\, dx$ on $\mathbb{R}[x]$ is a quadratic form.
   (d) The function $Q(A) = \det(A)$ on $M_{2\times 2}(\mathbb{R})$ is a quadratic form.
   (e) The function $Q(A) = \det(A)$ on $M_{3\times 3}(\mathbb{R})$ is a quadratic form.
   (f) Every quadratic form over $\mathbb{R}$ is a bilinear form.
   (g) Every quadratic form over an arbitrary field is a bilinear form.
   (h) The second derivatives test classifies any critical point as a local minimum, local maximum, or saddle.
   (i) If both eigenvalues of the 2×2 real symmetric matrix S are positive, then the graph of $(x, y)\cdot S\cdot(x, y)^T = 1$ in $\mathbb{R}^2$ will be an ellipse.
   (j) If one eigenvalue of the 2×2 real symmetric matrix S is zero and the other is nonzero, then the graph of $(x, y) \cdot S \cdot (x, y)^T = 1$ in $\mathbb{R}^2$ will be a hyperbola.
   (k) The singular values of $T : V \to V$ are the absolute values of the eigenvalues of T.
   (l) If T is Hermitian, the singular values of $T : V \to V$ are absolute values of the eigenvalues of T.
   (m) The singular value decomposition of a matrix is unique.
   (n) If $T : V \to W$ is linear, the pseudoinverse $T^\dagger$ satisfies $T^\dagger T(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \operatorname{im}(T)$.
   (o) If $T : V \to W$ is linear, the pseudoinverse $T^\dagger$ satisfies $T^\dagger T(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in \operatorname{im}(T)^\perp$.

---

2. Consider the bilinear form $\Phi[(a, b), (c, d)] = 4ab - 2ad - 2bc + 7bd$ on $\mathbb{R}^2$ with associated quadratic form Q.
   (a) Write down Q explicitly and also find $[\Phi]_\beta$ for $\beta = \{(1, 0), (0, 1)\}$.
   (b) Find an orthonormal basis γ for $\mathbb{R}^2$ such that $[\Phi]_\gamma$ is diagonal, and compute the diagonalization $[\Phi]_\gamma$.
   (c) Describe the shape of the quadratic variety $Q(x, y) = 1$ in $\mathbb{R}^2$ as one of the 3 standard conic sections.
   (d) Classify the critical point of $Q(x, y)$ at $(0, 0)$ as a local minimum, local maximum, or saddle.
   (e) Calculate the signature and index of Q, and determine the definiteness of Q.

---

3. Consider the quadratic form $Q(x, y, z) = 11x^2 + 40xy - 16xz - 16y^2 - 16yz + 5z^2$ on $\mathbb{R}^3$.
   (a) Find the symmetric matrix S associated to the underlying bilinear form for Q with respect to the standard basis $\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
   (b) Give an explicit orthonormal change of basis that diagonalizes Q, and find the resulting diagonalization.
   (c) Describe the shape of the quadratic variety $Q(x, y, z) = 1$ in $\mathbb{R}^3$ as one of the 9 standard quadric surfaces.
   (d) Classify the critical point of $Q(x, y, z)$ at $(0, 0, 0)$ as a local minimum, local maximum, or saddle.
   (e) Calculate the signature and index of Q, and determine the definiteness of Q.

---

4. For each matrix M, find (i) the singular values of M, (ii) a singular value decomposition $M = U\Sigma V^*$ where U and V are unitary and Σ is a rectangular diagonal matrix, and (iii) the pseudoinverse $M^\dagger$ of M:

(a) $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 2 \\ 2 & 1 \\ 3 & 6 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & i & -1 & -i \\ 2 & 2 & 2 & 2 \end{bmatrix}$

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. In multivariable calculus, the following more explicit version of the second derivative test is often taught[1]:

- Theorem (Second Derivatives Test in $\mathbb{R}^2$): Suppose P is a critical point of $f(x, y)$, and let D be the value of the discriminant $f_{xx}f_{yy} - f_{xy}^2$ at P. If $D > 0$ and $f_{xx} > 0$, then the critical point is a minimum. If $D > 0$ and $f_{xx} < 0$, then the critical point is a maximum. If $D < 0$, then the critical point is a saddle point. If $D = 0$, then the test is inconclusive.

Using our general version of the second derivatives test, prove this variation. [Hint: Note that $D = \det(H) = \lambda_1\lambda_2$; then examine what information the sign of D yields about the eigenvalues λ_1, λ_2.]

---

6. Let S be an $n \times n$ real symmetric matrix.

(a) Show that S is congruent to a matrix whose diagonal entries are all in the set $\{-1, 0, 1\}$.

(b) Prove that, up to congruence, there are exactly $\frac{1}{2}(n+1)(n+2)$ different real $n \times n$ symmetric matrices.

---

7. By the singular value decomposition theorem, if $T : V \to W$ is a linear transformation of rank r, then there exist orthonormal bases $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of V and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ of W along with scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ with $T(\mathbf{v}_i) = \sigma_i \mathbf{w}_i$ for $1 \leq i \leq r$ and $T(\mathbf{v}_i) = \mathbf{0}$ for $i > r$.

(a) Show that $T^*(\mathbf{w}_i) = \sigma_i \mathbf{v}_i$ for $1 \leq i \leq r$ and $T(\mathbf{w}_i) = \mathbf{0}$ for $i > r$. [Hint: Consider $[T]_\beta^\gamma$.]

(b) Show that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of eigenvectors for T^*T with corresponding eigenvalues $\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0$, and that $\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ is a set of eigenvectors for TT^* with corresponding eigenvalues $\sigma_1^2, \dots, \sigma_r^2, 0, \dots, 0$.

(c) Deduce that the nonzero eigenvalues of T^*T and TT^* are the same, and hence that the nonzero singular values of T and T^* are the same.

(d) Show that if $A \in M_{m\times n}(\mathbb{C})$, then the singular values of A and A^* are the same, and that A^* has a singular value decomposition $A = U\Sigma V^*$, then A^* has a singular value decomposition $A^* = V\Sigma^T U^*$.

- Remark: The results of this problem are useful in computing the SVD of a non-square matrix, since one may just find the nonzero eigenvalues and eigenvectors of the smaller of A^*A and $AA^*$to construct $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$, and then compute the kernel of A to get $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ and the kernel of A^* to get $\{\mathbf{w}_{r+1}, \dots, \mathbf{w}_m\}$.

---

8. Suppose $A \in M_{m\times n}(F)$ where $F = \mathbb{R}$ or $\mathbb{C}$.

(a) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, show that $(AB)^\dagger \neq B^\dagger A^\dagger$.

(b) Show that $(A^\dagger)^* = (A^*)^\dagger$.

(c) Show that $AA^\dagger$ and $A^\dagger A$ are positive-semidefinite and Hermitian.

---

[1]The statement of this theorem is copied directly from my multivariable calculus course notes, in fact!

9. [Optional] The goal of this problem is to discuss matrix square roots and the matrix analogue of the polar form $z = e^{i\theta}r$ of a complex number. Let $A \in M_{n\times n}(\mathbb{C})$.

   (a) Show that there exists a unitary matrix W and a positive semidefinite Hermitian matrix P such that $A = WP$; this is called a (right) polar decomposition of A, with W being the analogue of $e^{i\theta}$ and P being the analogue of r. [Hint: Take $W = UV^*$ and $P = V\Sigma V^*$.]

   (b) Show that if B is a positive-semidefinite Hermitian matrix such that $B^2 = \mu I_n$ for some nonnegative scalar μ, then $B = \sqrt{\mu} I_n$.

   (c) Show that if A is a positive-semidefinite Hermitian matrix, then there exists a unique positive-semidefinite Hermitian matrix B satisfying $B^2 = A$ (i.e., a "square root" of A). [Hint: Reduce to the case where A is diagonal, and then use part (b) along with 7(a) from homework 8 on each eigenspace of A.]

   (d) Suppose P and Q are positive-semidefinite Hermitian matrices and $P^2 = Q^2$. Show that $P = Q$.

   (e) Show that the polar decomposition of an invertible matrix A is unique. [Hint: Show first that P is invertible and then that $WP = ZQ$ implies $P^2 = Q^2$.]

   - Remark: The usual procedure for finding the polar form of a complex number $z = e^{i\theta}r$ is to note that $r = \sqrt{|z|^2} = \sqrt{\overline{z}z}$ and then $e^{i\theta} = z/r$. For the polar decomposition $A = WP$ we have an analogous formula: $P = \sqrt{A^*A}$, where the square root here denotes the positive-semidefinite matrix square root of (c), and when P is positive-definite we obtain the unitary part W via $W = AP^{-1}$.
