E. Dummit's Math 4571 ~ Advanced Linear Algebra, Spring 2022 ~ Homework 9, due Mon Apr 18th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Let V be a vector space with scalar field F and $\Phi : V \times V \to F$ be a bilinear form. Identify each of the following statements as true or false:

   (a) If $V = \mathbb{R}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ is the usual inner product on $\mathbb{R}^2$, then Φ is a bilinear form on V.

   (b) If $V = \mathbb{C}^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \overline{\mathbf{w}}$ is the usual inner product on $\mathbb{C}^2$, then Φ is a bilinear form on V.

   (c) If $V = \mathbb{R}$ and $\Phi(x, y) = x + 2y$, then Φ is a bilinear form on V.

   (d) If $V = F^2$ and $\Phi(\mathbf{v}, \mathbf{w}) = \det(\mathbf{v}, \mathbf{w})$, the determinant of the matrix whose columns are $\mathbf{v}$ and $\mathbf{w}$, then Φ is a bilinear form on V.

   (e) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \text{tr}(AB)$, then Φ is a bilinear form on V.

   (f) If $V = M_{n\times n}(F)$ and $\Phi(A, B) = \det(AB)$, then Φ is a bilinear form on V.

   (g) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 x f(x)g(x)\, dx$, then Φ is a bilinear form on V.

   (h) If $V = C[0, 1]$ and $\Phi(f, g) = \int_0^1 f'(x)g'(x)\, dx$, then Φ is a bilinear form on V.

   (i) If Φ is a symmetric bilinear form, then $[\Phi]_\beta$ is a symmetric matrix for any basis β.

   (j) If $[\Phi]_\beta$ is a symmetric matrix for some basis β, then Φ is a symmetric bilinear form.

   (k) If $\mathcal{B}(V)$ is the space of all bilinear forms on V and $\dim_F(V) = n$, then $\dim_F \mathcal{B}(V) = 2n$.

   (l) Congruent matrices have the same eigenvalues.

   (m) Congruent matrices have the same eigenvectors.

   (n) Every $n \times n$ symmetric matrix over $\mathbb{R}$ is congruent to a diagonal matrix.

   (o) Every $n \times n$ symmetric matrix over an arbitrary field F is congruent to a diagonal matrix.

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2. Solve each system of differential equations:

   (a) Find the general solution to $\left\{ \begin{array}{rcl} y_1' &=& 7y_1 + y_2 \\ y_2' &=& 9y_1 - y_2 \end{array} \right\}$.

   (b) Find the general solution to $\left\{ \begin{array}{rcl} y_1' &=& 3y_1 - 2y_2 \\ y_2' &=& y_1 + y_2 \end{array} \right\}$.

   (c) Find the general solution to $y'' - 4y = 0$. [Hint: Set $z = y'$ and convert to a system of linear equations.]

   (d) Find all functions y_1 and y_2 such that $\begin{array}{rcl} y_1' &=& 2y_2 + \sec(2x) \\ y_2' &=& -2y_1 \end{array}$.

   (e) Solve the system $\mathbf{y}'(t) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}$, where $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

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3. For each bilinear form on each given vector space, compute $[\Phi]_\beta$ for the given basis β:

   (a) The pairing $\Phi((a, b, c), (d, e, f)) = ad + ae - 2be + 3cd + cf$ on $V = F^3$ with β the standard basis.

   (b) The pairing $\Phi(p, q) = p(-1)q(2)$ on $V = P_2(\mathbb{R})$ with $\beta = \{1, x, x^2, x^3\}$.

   (c) The pairing $\Phi(A, B) = \text{tr}(AB)$ on $V = M_{2\times 2}(\mathbb{C})$ with $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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4. For each symmetric matrix S over each given field, find an invertible matrix Q and diagonal matrix D such that $Q^T SQ = D$:

(a) $S = \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix}$ over $\mathbb{Q}$. (b) $S = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & 6 \\ -2 & 6 & 7 \end{bmatrix}$ over $\mathbb{Q}$. (c) $S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ over $\mathbb{Q}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

5. For $A, B \in M_{n\times n}(F)$, recall that we say A is congruent to B when there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that $B = Q^T AQ$. Prove that congruence is an equivalence relation on $M_{n\times n}(F)$.

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6. Suppose $T : V \to V$ is a linear operator on the real inner product space V with inner product $\langle \cdot, \cdot \rangle$. Define the map $\Phi : V \times V \to F$ by setting $\Phi(\mathbf{v}, \mathbf{w}) = \langle T(\mathbf{v}), \mathbf{w}\rangle$.

   (a) Show that Φ is a bilinear form on V.
   (b) Show that Φ is symmetric if and only if T is Hermitian.
   (c) If V is finite-dimensional, prove that Φ is an inner product on V if and only if T is positive-definite and Hermitian. [Hint: Show that [I3] requires all eigenvalues of T to be positive.]

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7. Suppose V is finite-dimensional and $T : V \to V$. We say the polynomial $q(x) \in F[x]$ <u>annihilates</u> T if $q(T) = 0$.

   (a) Show that the set of polynomials in $F[x]$ annihilating T is a vector space.

   We define the <u>minimal polynomial</u> of T to be the monic polynomial $m(t) \in F[t]$ of smallest positive degree annihilating T. For example, the minimal polynomial of the identity transformation is $m(t) = t - 1$.

   (b) Show that every polynomial that annihilates T is divisible by the minimal polynomial. [Hint: Use polynomial division.]
   (c) Conclude that the minimal polynomial divides the characteristic polynomial.
   (d) Suppose λ is an eigenvalue of T. Prove that λ is a root of the minimal polynomial of T, and deduce that the minimal polynomial and the characteristic polynomial have the same roots. [Hint: Consider the Jordan form of an associated matrix A.]
   (e) Parts (c) and (d) gives a moderately effective way to find the minimal polynomial, namely, test divisors of the characteristic polynomial that have all of the same roots. Using this method or otherwise, find the minimal polynomials of the matrices $\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & -1 & 3 \end{bmatrix}$.
   (f) Show that similar matrices have the same minimal polynomial.
   (g) Show that the minimal polynomial of the $k \times k$ Jordan block with eigenvalue λ is $m(t) = (t - \lambda)^k$.
   (h) Show that the exponent of $t - \lambda$ in the minimal polynomial $m(t)$ of A is the size of the largest Jordan block of eigenvalue λ in the Jordan canonical form of A.
   (i) Show that a matrix is diagonalizable if and only if its minimal polynomial has no repeated roots.

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8. [Optional] A Hermitian matrix A is said to be <u>positive-definite</u> if $\mathbf{v}^* A\mathbf{v} > 0$ for every $\mathbf{v} \neq \mathbf{0}$. The goal of this problem is to prove Sylvester's criterion for positive-definiteness: if A is an $n \times n$ Hermitian matrix, then A is positive definite if and only if $\det A^{(k)} > 0$ for all $1 \le k \le n$, where $A^{(k)}$ is the upper $k \times k$ submatrix of A. So suppose $A \in M_{n\times n}(\mathbb{C})$ is Hermitian.

   (a) If A is positive definite, show that $A^{(k)}$ is positive definite for each $1 \le k \le n$ and deduce that $\det A^{(k)} > 0$ for all $1 \le k \le n$.
   (b) Suppose that A has two orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ whose eigenvalues λ_1, λ_2 are negative. Show that $A^{(k-1)}$ is not positive definite. [Hint: Show that there exists a linear combination $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ whose last coordinate is zero, and then that $\mathbf{w}^* A\mathbf{w} < 0$.]
   (c) Deduce that if $A^{(k-1)}$ is positive definite and $\det(A) > 0$, then all eigenvalues of A must be positive and hence A is positive definite.
   (d) Suppose that $\det A^{(k)} > 0$ for all $1 \le k \le n$. Show that A is positive definite.
