Directions: Read ALL of the following directions.

This is an open-notes, open-homework, open-textbook exam. There is no official time limit, but it is suggested that you should be able to solve most of the problems within approximately 5 hours.

There are 16 problems on this exam, with points as indicated.

Justify any answers you give, including computations. You may freely refer to results from in class, from the course notes, course assignments and solutions, and the course textbook, but please make it clear what results you are using.

Proofs and explanations are expected to be clear, concise, and correct.

In problems with multiple parts, you MAY use the results of previous parts in later parts, even if you were unable to solve the earlier parts correctly.

You MAY use a computer for typesetting and to access any material on the course webpage (e.g., the course notes and homework solutions), and to perform computations. Any such computations must be clearly identified and justified as correct.

You MAY NOT use a computer to access any other information.

You MAY ask the instructor for help on any part of the exam, during office hours or via email, and in fact YOU ARE ENCOURAGED to do so.

You MAY NOT discuss the material on this exam with anyone except for the instructor (until after the due date). This includes asking others for hints or solutions, searching for information about the problems online, or posting about the problems on discussion forums.

Please include AND SIGN the following statement with your exam:

I certify that I have neither given nor received any assistance on this exam and have used no resources other than those allowed.

Exams submitted without this certification WILL NOT BE GRADED.

Are we going to have to think today, or is it going to be all math?

(Anonymous Student)

Arithmetic! Algebra! Geometry! Grandiose trinity! Whoever has not known you is without sense!

Comte de Lautreamont

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

Emil Artin

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

John von Neumann

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1. (10) Identify each of the following statements as true or false:

   (a) If W is a subspace of the finite-dimensional vector space V, then there exists a basis for V containing a basis for W.

   (b) If $\dim(V) = 7$, $\dim(W) = 5$, and $T : V \to W$ is onto, then $\dim(\operatorname{im} T) = 2$.

   (c) There exists a linear $T : \mathbb{R}^5 \to M_{2\times 2}(\mathbb{R})$ whose nullity and rank are equal.

   (d) If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, then $||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$.

   (e) The matrices $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ are similar over $\mathbb{C}$.

   (f) If a 6×6 real matrix has characteristic polynomial $p(t) = t^6 - t^2$, then the generalized 0-eigenspace has dimension 2.

   (g) The pairing $\Phi(A, B) = \operatorname{tr}(AB)$ is a bilinear form on $M_{5\times 5}(\mathbb{R})$.

   (h) Every symmetric real matrix is congruent over the real numbers to a diagonal matrix.

   (i) A quadratic form on $\mathbb{R}^8$ with index 4 and signature 4 is positive semidefinite.

   (j) Every matrix in $M_{n\times n}(\mathbb{C})$ is the product of a unitary matrix, a diagonal matrix, and a unitary matrix.

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2. (5) Consider the derivative map $D : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $D[f(x)] = f'(x)$.

   (a) Find the matrix $[D]_\beta^\beta$ of D with respect to the basis $\beta = \{1, x, x^2, x^3\}$.

   (b) Find the eigenvalues and a basis for each eigenspace of D.

   (c) Is D diagonalizable? What is its Jordan canonical form?

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3. (10) Suppose the characteristic polynomial of the 7×7 matrix A is $p(t) = (t + 1)^4(t - 2)^2(t - 3)$.

   (a) Find the eigenvalues of A, and list all possible dimensions for each of the corresponding eigenspaces.

   (b) Find the determinant and trace of A.

   (c) List all possible Jordan canonical forms of A up to equivalence.

   (d) Find the Jordan canonical form of A if $\operatorname{rank}(I + A) = 4$ and there are two linearly independent 2-eigenvectors of A.

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4. (10) Let $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$.

   (a) Find the eigenvalues of A and a basis for each eigenspace.

   (b) Find a real orthogonal matrix Q (i.e., with $Q^T = Q^{-1}$) such that $Q^{-1}AQ$ is diagonal.

   (c) Find a formula for the nth power A^n.

   (d) Solve the system of differential equations $\left\{ \begin{array}{rcl} y_1' & = & 3y_1 + 4y_2 \\ y_2' & = & 4y_1 - 3y_2 \end{array} \right\}$ using any method.

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5. (10) Consider the symmetric bilinear form $\Phi[(a,b,c),(d,e,f)] = 6ad - 2ae - 2af - 2bd + 5be - 2cd + 7cf$ on $\mathbb{R}^3$ and let $Q(x,y,z)$ be the associated quadratic form.

   (a) Find the matrix $S = [\Phi]_\beta$ with respect to the standard basis $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$.

   (b) Evaluate $Q(x,y,z)$ explicitly.

   (c) Give an explicit orthonormal basis γ of $\mathbb{R}^3$ such that the associated matrix $[\Phi]_\gamma$ is diagonal, and find the corresponding diagonalization.

   (d) Identify the surface $Q(x,y,z) = 1$ in $\mathbb{R}^3$ as one of the 9 standard quadric surfaces.

   (e) Classify the critical point of the function $Q(x,y,z)$ at $(0,0,0)$ as a local minimum, local maximum, or saddle point.

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6. (10) Let $A = \begin{bmatrix} 2 & -8 & 2 \\ 6 & 6 & -9 \end{bmatrix}$.

   (a) Find singular value decompositions for A and for A^T.

   (b) Find the pseudoinverses $A^\dagger$ and $(A^T)^\dagger$.

   (c) Find the solution $\mathbf{x}$ to the system $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ that has minimal norm.

   (d) Find the least-squares solution to the inconsistent system $A^T\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

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7. (5) Suppose $T : V \to W$ is a linear transformation where V and W are finite-dimensional and $\dim(V) > \dim(W)$. Prove that there must exist vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ but $\mathbf{v}_1 \neq \mathbf{v}_2$.

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8. (5) If W_1 and W_2 are subspaces of the finite-dimensional inner product space V, prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$. [Hint: The statements are equivalent by using $(W^\perp)^\perp = W$.]

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9. (10) Suppose V is an inner product space and $T : V \to V$ is linear. Recall that we say T is an <u>isometry</u> if $\langle T(\mathbf{v}), T(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in V$.

   (a) If T is an isometry, show that any eigenvalue $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$.

   (b) Conversely, suppose that V possesses an orthonormal basis of eigenvectors for T each of whose eigenvalues λ has $|\lambda| = 1$. Prove that T is an isometry. [Hint: Write $\mathbf{v}, \mathbf{w}$ in terms of this basis and compute $\langle T(\mathbf{v}), T(\mathbf{w}) \rangle$.]

   (c) If V is finite-dimensional and T is an isometry that is also Hermitian, show that T^2 is the identity map and that T is unitary. [Hint: Consider the diagonalization.]

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10. (10) A matrix $A \in M_{n\times n}(\mathbb{C})$ is called <u>nilpotent</u> if A^k is the zero matrix for some positive integer k.

   (a) If A is diagonalizable and nilpotent, show that it is the zero matrix.

   (b) If A is nilpotent, show that the only eigenvalue of A is $\lambda = 0$.

   (c) Conversely, if the only eigenvalue of A is $\lambda = 0$, show that A is nilpotent. [Hint: Consider the characteristic polynomial.]

   (d) Find all possible Jordan canonical forms for a 4×4 nilpotent matrix.

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11. (5) Suppose $A \in M_{n\times n}(\mathbb{C})$. Prove that $\det(e^A) = e^{\mathrm{tr}(A)}$, and deduce that e^A is always invertible. [Hint: Put A in Jordan form.]

Remark: In fact, the inverse of e^A is e^{-A}.

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12. (5) Let V be a (not necessarily finite dimensional) vector space with scalar field F, with $c \in F$, and suppose that Φ_1 and Φ_2 are two bilinear forms on V.

   (a) Show that $\Phi_1 + \Phi_2$ is a bilinear form on V. Note that $\Phi_1 + \Phi_2$ is defined pointwise, so $(\Phi_1 + \Phi_2)(\mathbf{v}, \mathbf{w}) = \Phi_1(\mathbf{v}, \mathbf{w}) + \Phi_2(\mathbf{v}, \mathbf{w})$.

   (b) Show that $c\Phi_1$ is a bilinear form on V.

   (c) Show that $(\Phi_1 + c\Phi_2)^T = \Phi_1^T + c\Phi_2^T$. Deduce that if Φ_1 and Φ_2 are symmetric then so is $\Phi_1 + c\Phi_2$.

   (d) Show that the set of all bilinear forms on V is a vector space, and that the space of symmetric bilinear forms is a subspace of it.

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13. (10) Let A, B be elements of $M_{n\times n}(F)$ for an arbitrary field F.

   (a) If A and B are similar over F, prove that A^T and B^T are also similar over F.

   (b) If A and B are congruent over F, prove that A^T and B^T are also congruent over F.

   (c) If A and B are similar over F and invertible, prove that A^{-1} and B^{-1} are also similar over F.

   (d) If A is invertible and diagonalizable over F, prove that A^{-1} is also diagonalizable over F.

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14. (5) Suppose $A \in M_{m\times n}(\mathbb{C})$. Show that $I_n + A^*A$ is positive definite. [Hint: Show A^*A is positive-semidefinite.]

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15. (5) Suppose $A \in M_{m\times n}(\mathbb{C})$. If W is any $m \times m$ unitary matrix, show that the pseudoinverse $(WA)^\dagger$ equals $A^\dagger W^*$.

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16. (10) Prove or disprove the following statement:

“Linear algebra is a boring and useless subject with no practical applications.”
