E. Dummit's Math 3527 ~ Number Theory I, Spring 2022 ~ Homework 8, due Fri Apr 1st.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Use the Euclidean algorithm in each Euclidean domain to compute a greatest common divisor of each pair of elements, and then to write it as a linear combination of the elements:

   (a) The polynomials $x^6 - 1$ and $x^8 - 1$ in $\mathbb{R}[x]$.

   (b) The elements $11 + 27i$ and $-9 + 7i$ in $\mathbb{Z}[i]$.

   (c) The polynomials $x^3 + x^2 + 1$ and $x^4 + x$ in $\mathbb{R}[x]$.

   (d) The elements $43 - i$ and $50 - 50i$ in $\mathbb{Z}[i]$.

   (e) The elements $x^4 + 2x + 1$ and $x^3 + x$ in $\mathbb{F}_3[x]$.

   (f) The elements $9 + 43i$ and $22 + 10i$ in $\mathbb{Z}[i]$.

---

2. Let $R = \mathbb{F}_3[x]$ and $p = x^2 + x$.

   (a) List the 9 residue classes in R/pR.

   (b) Construct the addition and multiplication tables for R/pR. (You may omit the bars in the residue class notation for efficiency.)

   (c) Identify all of the units and zero divisors in R/pR.

---

3. Let $R = \mathbb{F}_2[x]$ and $p = x^3 + x + 1$.

   (a) List the 8 residue classes in R/pR.

   (b) Construct the addition and multiplication tables for R/pR. (You may omit the bars in the residue class notation for efficiency.)

   (c) Show that R/pR is a field by explicitly identifying the inverse of every nonzero element. [Hint: Use the multiplication table from (b).]

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Show the following things:

   (a) Show that the element $4 + 5i$ is irreducible and prime in $\mathbb{Z}[i]$.

   (b) Show that the element $x^2 + 4x + 5$ is irreducible and prime in $\mathbb{R}[x]$.

   (c) Show that the element $x^2 + 4x + 5$ is neither irreducible nor prime in $\mathbb{C}[x]$ by finding a factorization.

   (d) Show that the element $3 + 5i$ is neither irreducible nor prime in $\mathbb{Z}[i]$ by finding a factorization.

   (e) Show that the element $2 + \sqrt{-10}$ is irreducible but not prime in $\mathbb{Z}[\sqrt{-10}]$. [Hint: Show it divides 14 and that there are no elements of norm 2 or 7.]

---

5. The goal of this problem is to prove that the ring $R = \mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain under its norm function $N(a + b\sqrt{-2}) = a^2 + 2b^2$ using a similar argument to the one used to show $\mathbb{Z}[i]$ is Euclidean.

   (a) Suppose that $c + d\sqrt{-2}$ is not zero. Write $\dfrac{a + b\sqrt{-2}}{c + d\sqrt{-2}}$ in the form $x + y\sqrt{-2}$ for rational numbers x and y. [Hint: Rationalize the denominator.]

   (b) With notation from part (a), let s be the closest integer to x and t be the closest integer to y. Set $q = s + t\sqrt{-2}$ and $r = (a + b\sqrt{-2}) - (s + t\sqrt{-2})(c + d\sqrt{-2})$. Prove that $N(r) \le \frac{3}{4} N(c + d\sqrt{-2})$.

   (c) Deduce that R is a Euclidean domain.

   (d) Use the Euclidean algorithm in R to find the greatest common divisor of $33 + 5\sqrt{-2}$ and $8 + 11\sqrt{-2}$ in R, and then write it as a linear combination of these elements.

   - Remark: By a similar argument, $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$ are also Euclidean domains.
