E. Dummit's Math 3527 ~ Number Theory I, Spring 2022 ~ Homework 3, due Fri Feb 11th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each integer a and modulus m, determine whether the residue class $\overline{a}$ is a unit modulo m, or a zero divisor modulo m. If $\overline{a}$ is a unit then find its multiplicative inverse, while if $\overline{a}$ is a zero divisor then find a nonzero residue class $\overline{x}$ such that $\overline{a} \cdot \overline{x} = \overline{0}$.
   (a) $a = 14$, $m = 49$. (b) $a = 16$, $m = 49$. (c) $a = 1776$, $m = 2022$. (d) $a = 1789$, $m = 2022$.

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2. Find the general solution to each of the given congruences, or explain why there is no solution:
   (a) $4n + 3 \equiv 2 \pmod{19}$.
   (b) $3n \equiv 7 \pmod{21}$.
   (c) $3n \equiv 9 \pmod{21}$.
   (d) $1789n \equiv 1492 \pmod{2022}$.
   (e) $36n \equiv 114 \pmod{2022}$.

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3. Using the Chinese Remainder Theorem or otherwise, find the general solution n to each system of congruences:
   (a) $4n + 3 \equiv 2 \pmod{19}$.
   (b) $n \equiv 4 \pmod{11}$ and $n \equiv 1 \pmod{15}$.
   (c) $n \equiv 7 \pmod{999}$ and $n \equiv 37 \pmod{1001}$.
   (d) $n \equiv 7 \pmod{84}$ and $n \equiv 21 \pmod{35}$.
   (e) $n \equiv 7 \pmod{85}$ and $n \equiv 21 \pmod{34}$.
   (f) $n \equiv 2 \pmod{8}$, $n \equiv 1 \pmod{5}$, $n \equiv 3 \pmod{9}$.
   (g) $n \equiv 1 \pmod{44}$, $n \equiv 81 \pmod{90}$, and $n \equiv 61 \pmod{80}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. The goal of this problem is to discuss some applications of modular arithmetic to solving equations in integers (these are generally called Diophantine equations).
   (a) If n is a positive integer, show that n^2 is congruent to 0 or 1 modulo 4. [Hint: Square the four possible residue classes modulo 4.]
   (b) Show that there do not exist integers a and b such that $a^2 + b^2 = 2023$. [Hint: Work modulo 4.]
   (c) Strengthen (a) by showing that if n is a positive integer, then n^2 is congruent to 0, 1, or 4 modulo 8.
   (d) Show that there do not exist integers a, b, and c such that $a^2 + b^2 + c^2 = 2023$.

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5. Suppose n is a positive integer.
   (a) Show that $n^5 - n \equiv 0 \pmod{30}$. [Hint: By the Chinese Remainder Theorem, this is equivalent to showing $n^5 - n$ is divisible by 2, 3, and 5.]
   (b) Show that $n^8 - n^2 \equiv 0 \pmod{84}$.

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6. The goal of this problem is to establish the binomial theorem; for no additional charge, we will do this in an arbitrary commutative ring with 1. Define the binomial coefficient $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ for integers $0 \le k \le n$, and note that $\binom{n}{0} = \binom{n}{n} = 1$ for every n. (Recall the definition of $n!$ from homework 1.)
   (a) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for every $0 \le k \le n$. Conclude in particular that $\binom{n}{k}$ is always an integer.
   (b) Let R be a commutative ring with 1. If x and y are arbitrary elements of R, prove that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for any positive integer n. [Hint: Use induction on n.]
