E. Dummit's Math 3527 ~ Number Theory I, Spring 2022 ~ Homework 1, due Fri Jan 28th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Submit scans of your responses via Canvas.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each pair of integers (a, b), use the Euclidean algorithm to calculate their greatest common divisor $d = \gcd(a, b)$ and also to find integers x and y such that $d = ax + by$.
   (a) $a = 12$, $b = 44$.
   (b) $a = 2022$, $b = 20232$.
   (c) $a = 5567$, $b = 12445$.
   (d) $a = 233$, $b = 144$.

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2. Find the prime factorizations of 1001, 2021, 2022, 2021^{2022}, and 12345654321.

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3. It is sometimes claimed (occasionally in actual textbooks) that if $p_1, p_2, \ldots, p_k$ are the first k primes, then the number $n = p_1 p_2 \cdots p_k + 1$ used in Euclid's proof is always prime for any $k \geq 1$. Find a counterexample to this statement.

Part II: Solve the following problems. Justify all answers with rigorous, clear explanations.

4. Prove the following basic properties of divisibility (note that these properties are referred to, but not proven, in the course notes; you are expected to give the details of the proof!):
   (a) If a, b, c are integers such that $a|b$ and $b|c$, then $a|c$.
   (b) If a, b, c, x, y are integers where $a|b$ and $a|c$, then $a|(xb + yc)$.

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5. The Fibonacci-Virahanka numbers are defined as follows: $F_1 = F_2 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci-Virahanka sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...
   (a) Prove that $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ for every positive integer n. [Hint: Use induction.]
   (b) Prove that $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ for every positive integer n.
   (c) Prove that $F_{n+1}^2 - F_n F_{n+2} = (-1)^n$ for every positive integer n.

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6. Prove that $\log_3 7$ is irrational. [Hint: Suppose otherwise, so that $\log_3 7 = a/b$. Convert this to statement about positive integers and find a contradiction.]

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7. Recall that the factorial of n is defined as $n! = n \cdot (n-1) \cdot \cdots \cdot 1$, so for example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. (Note that $0!$ is defined to be 1.)
   (a) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for every positive integer n.
   (b) Prove that $n! + 1$ and $(n+1)! + 1$ are relatively prime for every positive integer n. [Hint: Subtract $(n+1)! + 1$ from a multiple of $n! + 1$.]
   (c) If $n \geq 3$, prove that the integers $n! + 2$, $n! + 3$, ... , $n! + n$ are all composite. Deduce that there are arbitrarily large "prime gaps" (i.e., differences between consecutive prime numbers).
