E. Dummit's Math 3527 ~ Number Theory 1, Spring 2022 ~ Final Exam Part B Review Problems

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Topics on Part B of the final exam:

- Primitive roots modulo m
- Residue classes in $\mathbb{Z}[i]$ mod β
- Factorization in $\mathbb{Z}[i]$
- Sums of two squares, Pythagorean triples
- Quadratic residues and nonresidues
- Legendre symbols, Euler's criterion
- Quadratic reciprocity and applications
- Jacobi symbols and their applications

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1. Determine / calculate / find the following:

   (a) Determine whether there exists a primitive root modulo (each of) 34, 35, 36, and 37.
   (b) Find a primitive root modulo 3^{2022} and the total number of primitive roots modulo 3^{2022}.
   (c) Find a primitive root modulo $2 \cdot 3^{2022}$ and the total number of primitive roots modulo $2 \cdot 3^{2022}$.
   (d) Find the number of residue classes in $\mathbb{Z}[i]$ modulo $7 - 5i$.
   (e) Find a fundamental region and list of residue class representatives for $\mathbb{Z}[i]$ modulo $2 - i$.
   (f) Find the prime factorization of $5 + 5i$ in $\mathbb{Z}[i]$.
   (g) Find the prime factorization of $11 + 12i$ in $\mathbb{Z}[i]$.
   (h) Find the prime factorization of 999 in $\mathbb{Z}[i]$.
   (i) Determine which of 104, 224, 420, and 666 can be written as the sum of two squares.
   (j) Find two different ways of writing the integer $260 = 2^2 \cdot 5 \cdot 13$ as a sum of two squares.
   (k) Find two different ways of writing the integer $450 = 2 \cdot 3^2 \cdot 5^2$ as a sum of two squares.
   (l) Find four Pythagorean right triangles with a hypotenuse of length 65.
   (m) Find two Pythagorean right triangles with a leg of length 49.
   (n) List the quadratic residues modulo 19.
   (o) Find the number of quadratic residues modulo 43, modulo 49, and modulo 51.
   (p) Determine whether 7, 11, and 14 are quadratic residues modulo 43.
   (q) Determine whether 7, 11, and 14 are quadratic residues modulo 43^{2022}.
   (r) Determine whether 13 and 26 are quadratic residues modulo the prime 2027.
   (s) Determine whether 28 and 15 are quadratic residues modulo the prime 71.
   (t) Determine whether 7 is a quadratic residue modulo $143 = 11 \cdot 13$.
   (u) Calculate the Legendre symbols $\left(\frac{103}{307}\right)$ and $\left(\frac{141}{307}\right)$.
   (v) Calculate the Jacobi symbols $\left(\frac{47}{245}\right)$ and $\left(\frac{177}{245}\right)$.

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2. Solve the following:

   (a) Verify Euler's theorem for the residue class of $1 + i$ modulo $4 + i$ in $\mathbb{Z}[i]$.
   (b) Prove that there exists a solution to $x^2 \equiv 11 \pmod{97}$. Note 97 is prime.
   (c) Prove that there exists a solution to $x^2 + 6x \equiv 14 \pmod{101}$. Note 101 is prime.
   (d) Prove that there exists a solution to $x^2 + 6x \equiv 14 \pmod{101^2}$. Note 101 is prime.
   (e) If $p > 3$ is a prime, prove that 3 is a quadratic residue modulo p if and only if $p \equiv 1, 11 \pmod{12}$.
   (f) If $p > 3$ is a prime, prove that -3 is a quadratic residue modulo p if and only if $p \equiv 1 \pmod{3}$.
   (g) Characterize the primes dividing an integer of the form $n^2 + 4n + 1$, for n an integer.
   (h) Characterize the primes dividing an integer of the form $n^2 + 4n - 1$, for n an integer.
