1. For more detailed solutions to problems like these, see the homework assignments and lecture notes.

   (a) There are primitive roots mod 34 and 37 but not mod 35 or mod 36.

   (b) 2 is a primitive root mod 3^2 hence mod 3^{2022}. Total number is $\varphi(\varphi(3^{2022})) = 2 \cdot 3^{2020}$.

   (c) 2 is a primitive root mod 3^{2022} so $2 + 3^{2022}$ is a prim root mod $2 \cdot 3^{2022}$. Total number is $\varphi(\varphi(2 \cdot 3^{2022})) = 2 \cdot 3^{2020}$.

   (d) The number of residue classes is $N(7 - 5i) = 49 + 25 = 74$.

   (e) By drawing the fundamental region (square with vertices 0, β, $i\beta$, $(1+i)\beta = 0$, $2 - i$, $1 + 2i$, $3 + i$), and picking inequivalent points, we get 0, 1, 2, $1 + i$, $2 + i$.

   (f) We have $5 + 5i = (1 + i)(2 + i)(2 - i)$, up to associates.

   (g) We have $11 + 12i = i(2 - i)(7 - 2i)$, up to associates.

   (h) We have $999 = 3^3(6 - i)(6 + i)$, up to associates.

   (i) By Fermat's theorem, $104 = 10^2 + 2^2$ and $666 = 21^2 + 15^2$ can, 224 and 420 cannot.

   (j) Since $N(1 + i) = 2$, $N(2 \pm i) = 5$, $N(3 \pm 2i) = 13$, take $(1 + i)^2(2 + i)(3 + 2i) = -14 + 8i$ yielding $260 = 8^2 + 14^2$, and also $(1 + i)^2(2 + i)(3 - 2i) = 2 + 16i$ yielding $260 = 2^2 + 16^2$.

   (k) Since $N(1+i) = 2$, $N(3) = 3^2$, $N(2 \pm i) = 5$, take $(1+i)3(2+i)^2 = 21 - 3i$ yielding $450 = 21^2 + 3^2$, and also $(1 + i)3(2 + i)(2 - i) = 15 + 15i$ yielding $450 = 15^2 + 15^2$.

   (l) Solving $k(s^2 + t^2) = 65$ gives various cases: $k = 1$ and $s^2 + t^2 = 65$ (with $(s, t) = (8, 1)$ or $(7, 4)$), $k = 5$ with $s^2 + t^2 = 13$ (with $(s, t) = (3, 2)$), $k = 13$ with $s^2 + t^2 = 5$ (with $(s, t) = (2, 1)$). Yields four triangles $(2kst, k(s^2 - t^2), k(s^2 + t^2))$: 16-63-65, 25-60-65, 33-56-65, 39-52-65.

   (m) Solving $k(s^2 - t^2) = 49$ gives various cases: $k = 7$ with $(s + t)(s - t) = 7$ so $s = 4$ and $t = 3$, or $k = 1$ with $(s + t)(s - t) = 49$ giving $s + t = 49$, $s - t = 1$ so $s = 50, t = 49$. Yields two triangles $(2kst, k(s^2 - t^2), k(s^2 + t^2))$: 49-1200-1201, 49-168-175.

   (n) $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2 \equiv 1, 4, 9, 16, 6, 17, 11, 7, 5 \bmod 19$.

   (o) Mod 43 there are $(43-1)/2 = 21$, mod 49 the quadratic residues are the same as those mod 7 ($(7-1)/2 = 3$ choices) for a total of $7 \cdot 3 = 21$, mod $51 = 3 \cdot 17$ the quadratic residues are those that are QRs mod 3 (1 choice) and 17 (8 choices) for a total of $1 \cdot 8 = 8$.

   (p) Compute $\left(\frac{7}{43}\right) = -\left(\frac{43}{7}\right) = -\left(\frac{1}{7}\right) = -1$, $\left(\frac{11}{43}\right) = -\left(\frac{43}{11}\right) = -\left(\frac{-1}{11}\right) = 1$, and $\left(\frac{14}{43}\right) = \left(\frac{2}{43}\right)\left(\frac{7}{43}\right) = (-1)(-1) = 1$ since $\left(\frac{2}{p}\right) = -1$ for $p \equiv 3, 5 \bmod 8$. So 11 and 14 are QRs mod 43 but 7 is not.

   (q) The QRs mod 43^{2022} are the same as those mod 43, so 11 and 14 are QRs but 7 is not.

   (r) Compute $\left(\frac{13}{2027}\right) = \left(\frac{2027}{13}\right) = \left(\frac{-1}{13}\right) = 1$ and $\left(\frac{26}{2027}\right) = \left(\frac{2}{2027}\right)\left(\frac{13}{2027}\right) = (-1)(1) = -1$ since $\left(\frac{2}{p}\right) = -1$ for $p \equiv 3, 5 \bmod 8$. So 13 is a QR but 26 is not.

   (s) Compute $\left(\frac{28}{71}\right) = \left(\frac{2}{71}\right)^2\left(\frac{7}{71}\right) = 1 \cdot -\left(\frac{71}{7}\right) = -\left(\frac{1}{7}\right) = -1$ and $\left(\frac{15}{71}\right) = -\left(\frac{71}{15}\right) = -\left(\frac{11}{15}\right) = \left(\frac{15}{11}\right) = \left(\frac{4}{11}\right) = 1$ using reciprocity for Jacobi symbols. So 15 is a QR but 28 is not.

   (t) We compute $\left(\frac{7}{11}\right) = -\left(\frac{11}{7}\right) = -\left(\frac{4}{7}\right) = -1$. Since this is -1, 7 is not a QR mod 11, and thus it also is not a QR mod 143. (Note however that the Jacobi symbol $\left(\frac{7}{143}\right) = +1$, even though 7 is not a QR.)

(u) We compute $\left(\frac{103}{307}\right) = -\left(\frac{307}{103}\right) = -\left(\frac{-2}{131}\right) = 1$ since $\left(\frac{-2}{p}\right) = -1$ for $p \equiv 5, 7 \bmod 8$, and $\left(\frac{141}{307}\right) = \left(\frac{307}{141}\right) = \left(\frac{25}{141}\right) = 1$.

(v) We compute $\left(\frac{47}{245}\right) = \left(\frac{245}{47}\right) = \left(\frac{10}{47}\right) = \left(\frac{2}{47}\right)\left(\frac{5}{47}\right) = 1 \cdot \left(\frac{47}{5}\right) = 1 \cdot \left(\frac{2}{5}\right) = -1$ since $\left(\frac{2}{p}\right) = 1$ for $p \equiv 1, 7 \bmod 8$, and $\left(\frac{177}{245}\right) = \left(\frac{245}{177}\right) = \left(\frac{68}{177}\right) = \left(\frac{2}{177}\right)^2\left(\frac{17}{177}\right) = \left(\frac{177}{17}\right) = \left(\frac{7}{17}\right) = \left(\frac{17}{7}\right) = \left(\frac{3}{7}\right) = -1$.

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2. Many problems of similar types were covered on the homework.

(a) First note that there are $N(4+i) = 17$ residue classes and since $4+i$ is irreducible, there are 16 units. Then $(1+i)^2 \equiv 2i$, so $(1+i)^4 \equiv (2i)^2 \equiv -4 \equiv i$, $(1+i)^8 \equiv i^2 \equiv -1$, and finally $(1+i)^{16} \equiv (-1)^2 \equiv 1$ as required.

(b) We compute $\left(\frac{11}{97}\right) = \left(\frac{97}{11}\right) = \left(\frac{9}{11}\right) = +1$, so the Legendre symbol is $+1$. This means 11 is a quadratic residue mod 97 so $x^2 \equiv 11 \pmod{97}$ has a solution.

(c) Completing the square gives $(x+3)^2 \equiv 5 \pmod{101}$ so we must determine whether 5 is a quadratic residue modulo 101. We compute $\left(\frac{5}{101}\right) = \left(\frac{101}{5}\right) = \left(\frac{1}{5}\right) = 1$, so 5 is a quadratic residue and thus there are solutions.

(d) As in (c) we have $(x+3)^2 \equiv 5 \pmod{101^2}$. The quadratic residues modulo 101^2 are the same as those mod 101, so since 5 is a QR mod 101 from (c), it is also a QR mod 101^2, so there is a solution here also.

(e) We want to compute $\left(\frac{3}{p}\right)$. If $p \equiv 1 \pmod 4$, then $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = +1$ only when $p \equiv 1 \pmod 3$ which together say $p \equiv 1 \pmod{12}$. Likewise, if $p \equiv 3 \pmod 4$, then $\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right) = +1$ only when $p \equiv 2 \pmod 3$, which together say $p \equiv 11 \pmod{12}$. If $p \equiv 5, 7 \pmod{12}$ then the calculations show $\left(\frac{3}{p}\right) = -1$.

(f) We want to compute $\left(\frac{-3}{p}\right)$. If $p \equiv 1 \pmod 4$, then $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = +1 \cdot \left(\frac{p}{3}\right) = +1$ only when $p \equiv 1 \pmod 3$. Likewise, if $p \equiv 3 \pmod 4$, then $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = -1 \cdot -\left(\frac{p}{3}\right) = \left(\frac{p}{3}\right) = +1$ only when $p \equiv 1 \pmod 3$. So in either case, $\left(\frac{-3}{p}\right) = +1$ only when $p \equiv 1 \pmod 3$.

(g) Completing the square gives $n^2 + 4n - 1 = (n+2)^2 - 5$, so we want primes p such that there is a solution to $(n+2)^2 \equiv 5 \pmod p$, which is equivalent to solving $x^2 \equiv 5 \pmod p$. Clearly there is a solution for $p = 2, 5$. For other p we compute $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$ which is $+1$ for $p \equiv 1, 4 \pmod 5$ and -1 for $p \equiv 2, 3 \pmod 5$. So p divides some $n^2 + 4n - 1$ iff $p = 2, 5$ or $p \equiv 1, 4 \pmod 5$.

(h) Completing the square gives $n^2 + 6n + 11 = (n+3)^2 + 2$, so we want primes p such that there is a solution to $(n+3)^2 \equiv -2 \pmod p$, which is equivalent to solving $x^2 \equiv -2 \pmod p$. Clearly there is a solution for $p = 2$. For other p we know $\left(\frac{-2}{p}\right) = +1$ precisely when $p \equiv 1, 3 \pmod 8$. So p divides some $n^2 + 6n + 11$ iff $p = 2$ or $p \equiv 1, 3 \pmod 8$.
