Math 4527 (Number Theory 2)

Lecture #6 of 38 ~ February 1, 2021

Rational Approximation and Transcendence

- Periodic Continued Fractions (wrapup)
- Rational Approximation via Continued Fractions
- Irrationality and Transcendence

This material represents §6.2.4-§6.2.5 from the course notes.

Continuation of Continued Fractions Continued, I

Theorem (Quadratic Irrationals and Continued Fractions)

Let α be a quadratic irrational with discriminant D and α_n be the nth remainder term from the continued fraction expansion of α.

1. *The remainder term α_n has discriminant D for all $n \geq 1$.*
2. *If α is a reduced quadratic irrational, then α_n is also reduced.*
3. *There are only finitely many reduced quadratic irrationals of discriminant D.*
4. *The remainder term α_n is reduced for sufficiently large n.*
5. *The continued fraction expansion of a real number α is periodic if and only if α is a quadratic irrational.*
6. *The continued fraction expansion of a real number α is purely periodic (i.e., is of the form $\alpha = [\overline{a_0, a_1, \dots, a_n}]$) if and only if α is a reduced quadratic irrational.*

Continuation of Continued Fractions Continued, II

6. The continued fraction expansion of a real number α is purely periodic (i.e., is of the form $\alpha = [\overline{a_0, a_1, \ldots, a_n}]$) if and only if α is a reduced quadratic irrational.

Proof:

- First suppose α has a purely periodic expansion.
- Then $\alpha = [a_0, a_1, \ldots, a_{kn}, \alpha]$ for every positive integer k. Since by (4) the remainders are eventually all reduced, this means α must be reduced.
- Conversely, suppose α is reduced. By (5) we know that the continued fraction expansion is eventually periodic, say with $\alpha_{k+n} = \alpha_k$ for some k and n.
- We will show that $\alpha_{k+n-1} = \alpha_{k-1}$. Then by iterating this fact, this implies $\alpha_{j+n} = \alpha_j$ for all $j \geq 0$.
- This immediately implies α has a periodic continued fraction expansion, because $a_{j+n} = \lfloor \alpha_{j+n} \rfloor = \lfloor \alpha_j \rfloor = a_j$ for all $j \geq 0$.

Continuation of Continued Fractions Continued, III

6. The continued fraction expansion of a real number α is purely periodic (i.e., is of the form $\alpha = [\overline{a_0, a_1, \ldots, a_n}]$) if and only if α is a reduced quadratic irrational.

Proof (continued):

- It remains to show that if α is reduced and $\alpha_{k+n} = \alpha_k$ then $\alpha_{k+n-1} = \alpha_{k-1}$. First, both α_{k+n} and α_k are reduced by (2).
- By definition we have $\alpha_{k+n} = \frac{1}{\alpha_{k+n-1} - a_{k+n-1}}$ and $\alpha_n = \frac{1}{\alpha_{n-1} - a_{n-1}}$, so conjugating and inverting yields $-\dfrac{1}{\overline{\alpha}_{n+k}} = a_{k+n-1} - \overline{\alpha}_{k+n-1}$ and $-\dfrac{1}{\overline{\alpha}_n} = a_{n-1} - \overline{\alpha}_{n-1}$.
- Since both $\overline{\alpha}_{k+n-1}$ and $\overline{\alpha}_{n-1}$ are between -1 and 0, we see $a_{k+n-1} = \lfloor -\dfrac{1}{\overline{\alpha}_{n+k}} \rfloor = \lfloor -\dfrac{1}{\overline{\alpha}_n} \rfloor = a_{n-1}$, as claimed.

Continuation of Continued Fractions Continued, IV

Example: Show that $(3 + \sqrt{13})/4$ is reduced and then find its continued fraction expansion.

Continuation of Continued Fractions Continued, IV

Example: Show that $(3+\sqrt{13})/4$ is reduced and then find its continued fraction expansion.

- Note $\alpha = (3+\sqrt{13})/4 > 1$ has $-1/\overline{\alpha} = 3+\sqrt{13} > 1$, so α is reduced. Per (6) in the proposition, its continued fraction expansion will be purely periodic.
- With $\alpha = (3+\sqrt{13})/4$, we find, successively,

n	0	1	2	3	4	5
α_n	$\frac{3+\sqrt{13}}{4}$	$\frac{1+\sqrt{13}}{3}$	$\frac{2+\sqrt{13}}{3}$	$\frac{1+\sqrt{13}}{4}$	$3+\sqrt{13}$	$\frac{3+\sqrt{13}}{4}$
a_n	1	1	1	1	6	
$\alpha_n - a_n$	$\frac{-1+\sqrt{13}}{4}$	$\frac{-2+\sqrt{13}}{3}$	$\frac{-1+\sqrt{13}}{3}$	$\frac{-3+\sqrt{13}}{4}$	$-3+\sqrt{13}$	

and we can see at this point each term will repeat. Therefore, the continued fraction expansion is $\boxed{[\overline{1,1,1,1,6}]}$, which is indeed periodic.

Rational Approximation and Continued Fractions, I

Our original motivation in developing continued fractions was to use them to give rational approximations of a real number α.

- We have already proven that if we compute the continued fraction expansion $\alpha = [a_0, a_1, a_2, \dots]$, then the successive convergents $C_n = p_n/q_n$ will converge to α as $n \to \infty$.
- In particular, these convergents will give better and better rational approximations to α.
- However, we will now prove some results that make the above heuristics far more precise: in fact we will show that (with suitable hypotheses) the continued fraction convergents are actually the *best possible* rational approximations to α.

Rational Approximation and Continued Fractions, II

Here are our main results:

Proposition (Rational Approximation and Continued Fractions)

Suppose α is any irrational real number and p/q is any rational number. Then the following hold:

1. *If p_n/q_n is the nth continued fraction convergent to α, and $\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{p_n}{q_n}\right|$, then $q > q_n$.*
2. *In fact, if $|q\alpha - p| < |q_n\alpha - p_n|$, then $q \geq q_{n+1}$.*
3. *There are infinitely many distinct rational numbers p/q such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.*
4. *If p/q is a rational number such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$, then in fact p/q is a continued fraction convergent to α.*

Rational Approximation and Continued Fractions, III

1. If p_n/q_n is the nth continued fraction convergent to α, and $\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{p_n}{q_n}\right|$, then $q > q_n$.

<u>Proof</u>:

- Consider the Farey sequence of level q_n: since $|p_{n-1}q_n - p_nq_{n-1}| = 1$, by our results on the Farey sequences we see that p_{n-1}/q_{n-1} and p_n/q_n are consecutive in this sequence.
- Hence, there is no rational number with denominator less than q_n that lies between them.
- Since α is between p_{n-1}/q_{n-1} and p_n/q_n, this means any rational approximation p/q that is closer than either of these must have denominator greater than q_n.

Rational Approximation and Continued Fractions, IV

2. In fact, if $|q\alpha - p| < |q_n\alpha - p_n|$, then $q \geq q_{n+1}$.

Proof:

- Suppose that $q < q_{n+1}$. By basic linear algebra, there exist integers x and y such that $p = xp_n + yp_{n+1}$ and $q = xq_n + yq_{n+1}$. (They are integers because the determinant $p_nq_{n+1} - p_{n+1}q_n$ of the associated coefficient matrix is ± 1 by our results on the convergents of the continued fraction.)
- Notice that since $q < q_{n+1}$, the second equation requires that one of x, y be positive and the other is negative. Since $\alpha - \dfrac{p_n}{q_n}$ and $\alpha - \dfrac{p_{n+1}}{q_{n+1}}$ also have opposite signs, this means $x\left(\alpha - \dfrac{p_n}{q_n}\right)$ and $y\left(\alpha - \dfrac{p_{n+1}}{q_{n+1}}\right)$ have the same sign.

Rational Approximation and Continued Fractions, V

2. In fact, if $|q\alpha - p| < |q_n\alpha - p_n|$, then $q \geq q_{n+1}$.

Proof (continued):

- Since $x\left(\alpha - \frac{p_n}{q_n}\right)$ and $y\left(\alpha - \frac{p_{n+1}}{q_{n+1}}\right)$ have the same sign, we can then write

$$\begin{aligned}
|q\alpha - p| &= |(xq_n + yq_{n+1})\alpha - (xp_n + yp_{n+1})| \\
&= |x(q_n\alpha - p_n) + y(q_{n+1}\alpha - p_{n+1})| \\
&= |x| \cdot |q_n\alpha - p_n| + |y| \cdot |q_{n+1}\alpha - p_{n+1}| \\
&\geq |q_n\alpha - p_n|
\end{aligned}$$

- Since we started by assuming that $q < q_{n+1}$, we have established the contrapositive of the desired result.

Rational Approximation and Continued Fractions, VI

3. There are infinitely many distinct rational numbers p/q such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.

Proof:

- We claim that at least one of p_n/q_n and p_{n+1}/q_{n+1} satisfies the desired inequality, so suppose that neither does.
- Then since α lies between p_n/q_n and p_{n+1}/q_{n+1}, we have

$$\begin{aligned}
\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right|^2 &= \left(\left|\frac{p_n}{q_n} - \alpha\right| + \left|\frac{p_{n+1}}{q_{n+1}} - \alpha\right|\right)^2 \\
&> 4\left|\frac{p_n}{q_n} - \alpha\right| \cdot \left|\frac{p_{n+1}}{q_{n+1}} - \alpha\right| \\
&\geq 4 \cdot \frac{1}{2q_n^2} \cdot \frac{1}{2q_{n+1}^2} = \frac{1}{q_n^2 q_{n+1}^2}.
\end{aligned}$$

Rational Approximation and Continued Fractions, VII

3. There are infinitely many distinct rational numbers p/q such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.

Proof (continued):

- In the middle step we used the inequality $(x+y)^2 \geq 4xy$, which is equivalent to $(x-y)^2 \geq 0$, and equality cannot hold in our case because α is irrational.
- Finally, taking the square root gives $\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| > \frac{1}{q_n q_{n+1}}$, but this is false since these quantities are equal.

Rational Approximation and Continued Fractions, VIII

4. If p/q is a rational number such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$, then in fact p/q is a continued fraction convergent to α.

Proof:

- Suppose p/q is not a convergent but that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.
- Let n be such that $q_n \le q < q_{n+1}$.
- By (2), we see $|q_n\alpha - p_n| < |q\alpha - p| = q\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q}$, so $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2qq_n}$.
- Now we get $\frac{1}{qq_n} \le \left|\frac{p_nq - pq_n}{qq_n}\right| = \left|\frac{p}{q} - \frac{p_n}{q_n}\right| \le \left|\frac{p}{q} - \alpha\right| + \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2qq_n} + \frac{1}{2q_n^2}$.
- But this implies $q < q_n$, which is a contradiction.

Rational Approximation and Continued Fractions, X

A few remarks about these results:

- The first statement (if $|\alpha - p/q| < |\alpha - p_n/q_n|$, then $q > q_n$) says that the best rational approximation to α, among all terms in the Farey sequence of level q_n, is the convergent p_n/q_n.
- Thus, if we want to find a good rational approximation of a given real number with small denominator, we can use convergents from the continued fraction expansion.
- Also, in the third statement (giving infinitely many p/q such that $|\alpha - p/q| < 1/(2q^2)$), the constant 2 is not sharp. It is a theorem of Hurwitz that the 2 can be replaced by $\sqrt{5}$ (the idea is to look at three terms rather than just 2) but not by any larger constant.

Rational Approximation and Continued Fractions, XI

Example: Find a rational number of small denominator with decimal expansion $0.4614379084967\ldots$.

Rational Approximation and Continued Fractions, XI

Example: Find a rational number of small denominator with decimal expansion 0.4614379084967 . . . .

- We compute the continued fraction expansion of $\alpha = 0.4614379084967$, which is easy to do with a calculator or computer: we get $\alpha = [0, 2, 5, 1, 57, 1, 53354674, 4, 1, 1, 6, 4]$.

Rational Approximation and Continued Fractions, XI

Example: Find a rational number of small denominator with decimal expansion $0.4614379084967\ldots$.

- We compute the continued fraction expansion of $\alpha = 0.4614379084967$, which is easy to do with a calculator or computer: we get $\alpha = [0, 2, 5, 1, 57, 1, 53354674, 4, 1, 1, 6, 4]$.
- We truncate just before the huge term in the middle to get a guess of $\alpha = [0, 2, 5, 1, 57, 1] = 353/765$. Indeed, we can calculate that $353/765 \approx 0.461437908496732$.

Rational Approximation and Continued Fractions, XII

Example: Find a rational number of small denominator with decimal expansion $0.4614379084967\ldots$.

- Indeed, because $\alpha = [0, 2, 5, 1, 57, 1, 53354674, 4, 1, 1, 6, 4]$, our other results indicate that any rational number that is a closer approximation will have denominator roughly on order of the next convergent
$$[0, 2, 5, 1, 57, 1, 53354674] = \frac{18834200269}{40816326362}.$$
- So the rational number $353/765$ we found is clearly the simplest!
- It is interesting to note that the period of the decimal expansion of $353/765$ is 16, so in fact we have identified the rational number before the expansion started repeating!

Rational Approximation and Continued Fractions, XIII

We can also use these results, along with some of our facts about the Farey sequences to find the best rational approximation to a given real number α having a denominator below a given fixed bound N.

- Our starting point is to calculate the last two convergents p_{n-1}/q_{n-1} and p_n/q_n whose denominators are less than N.
- Since the convergents alternate being above and below α, this means α lies between these two convergents.
- Furthermore, from the relation $|p_n q_{n-1} - p_{n-1} q_n| = 1$ and our results on the Farey sequences, we see that p_{n-1}/q_{n-1} and p_n/q_n are consecutive terms in the Farey sequence of level q_n.
- We can then generate all of the terms between these of level $\leq N$ by taking mediants, and from this short list we can identify the best approximation of α.

Rational Approximation and Continued Fractions, XIV

Example: Find the rational number with denominator less than 100 that is closest to $\sqrt{7}$.

- Earlier, we computed the continued fraction expansion $\sqrt{7} = [2, \overline{1, 1, 1, 4}]$.
- The first few convergents are then 2, 3, 5/2, 8/3, 37/14, 45/17, 82/31, 127/48, 590/223.
- The last two convergents with denominator less than 100 are 82/31 and 127/48. The only term between them in the Farey sequence of level 99 is their mediant, 209/79.
- We can then compute that $\sqrt{7} - 82/31 \approx 5.9 \cdot 10^{-4}$, $\sqrt{7} - 127/48 \approx -8.2 \cdot 10^{-5}$, and $\sqrt{7} - 209/79 \approx 1.8 \cdot 10^{-4}$. Thus, the best approximation is $\boxed{127/48}$.

Rational Approximation and Continued Fractions, XIV

<u>Example</u>: Find the rational number with denominator less than 10 that is closest to $\sqrt{7}$.

- From above, the last two convergents with denominator less than 10 are $5/2$ and $8/3$. The terms between them in the Farey sequence of level 9 are $18/7$, $13/5$, and $21/8$.
- We can then compute that $\sqrt{7} - 5/2 \approx 0.1458$, $\sqrt{7} - 18/7 \approx 0.0743$, $\sqrt{7} - 13/5 \approx 0.0458$, $\sqrt{7} - 21/8 \approx 0.0208$, and $\sqrt{7} - 8/3 \approx -0.0209$.
- Thus, the best approximation (by a quite small margin!) is $\boxed{21/8}$, which, we remark, is not a continued fraction convergent to $\sqrt{7}$.

Irrationality and Transcendence, I

We can use some of these properties of rational approximation we have developed to prove the irrationality of various quantities, and by suitably extending these results, we can even prove transcendence in some cases.

- One easy observation is that the continued fraction expansion of a real number α terminates in a finite number of steps if and only if α is rational. Thus, if the continued fraction of α is infinite, then α is irrational.
- We could therefore establish irrationality by computing the continued fraction expansion of a given real number.
- However, as a practical matter, this is not so easy to do. The easiest infinite continued fractions to compute are the periodic expansions, and as we proved, these are the expansions of quadratic irrationals. (But these numbers are quite easy to prove irrational.)

Irrationality and Transcendence, II

Another option, then, is to flip our approach around by instead constructing irrational numbers via their continued fraction expansions.

- For example, we automatically know that the real number $\alpha = [1, 2, 3, 4, 5, 6, \dots]$ is irrational, since its continued fraction expansion is not finite.
- However, an obvious problem arises: we have no simple way of giving a closed-form formula for this real number. (As it happens, this number can be written in terms of values of a modified Bessel function, though this is not so easy to prove.)
- As we have already seen, most numbers of interest to us do not have any obvious pattern to their continued fraction expansion, so this approach is also difficult.

Irrationality and Transcendence, III

Our second method is to use another of our earlier results: as we showed, if α is irrational, then there are infinitely many distinct rational numbers p/q such that $|\alpha - p/q| < 1/q^2$.

Our main idea is that the converse of this statement holds as well:

Proposition (Irrationality and Approximation)

A real number α is irrational if and only if there exist infinitely many distinct rational numbers p/q such that $|\alpha - p/q| < 1/q^2$.

Note that we already established the forward direction, so we just need to prove the reverse (if there are infinitely many such p/q then α is irrational).

Irrationality and Transcendence, IV

Proof:

- Suppose $\alpha = a/b$ is a fixed rational number.
- Then $|\alpha - p/q| = |aq - bp|/(bq)$. If $q \leq b$ then there are only finitely many possible p/q with $|\alpha - p/q| < 1/q^2$ since there are only finitely many possible denominators q and finitely many p that work for any given q.
- If $q > b$ then we would have $|aq - bp|/(bq) < 1/q^2$ so that $|aq - bp| < b/q < 1$. But since $|aq - bp|$ is an integer, it would then have to be zero, in which case p/q would equal a/b.
- Putting these two cases together shows that if α is rational, then there are only finitely many distinct rational numbers p/q such that $|\alpha - p/q| < 1/q^2$, as claimed.

Irrationality and Transcendence, V

In principle, we could try to use this result to establish the irrationality of an arbitrary irrational number. However, this can be quite cumbersome in practice.

- The numbers for which it will be effective are those that we can write as an infinite sum of rational numbers whose terms decrease rapidly in size: we can then obtain the desired rational approximations by taking partial sums of the series.
- As long as the tail of the series is very small (i.e., less than $1/q^2$) relative to the denominator q of each partial sum, we will be able to conclude that the sum of the series is irrational.

Irrationality and Transcendence, VI

Example: Show that $\alpha = \sum_{k=0}^{\infty} 10^{-3^k}$ is irrational.

- Let $p_n/q_n = \sum_{k=0}^{n} 10^{-3^k}$ be the nth partial sum of the series. We observe that $q_n = 10^{3^k}$ since each of the other terms has a denominator dividing 10^{3^k}.
- Furthermore, it is easy to see (e.g., from the decimal expansion of α) that the size of the tail $\sum_{k=n+1}^{\infty} 10^{-3^k}$ is at most $2 \cdot 10^{-3^{k+1}}$.
- Then we have an easy bound $|\alpha - p_n/q_n| < 2 \cdot 10^{-3^{k+1}} < (10^{-3^k})^2 = 1/q_n^2$. Since all of the partial sums of this series are distinct, we obtain infinitely many such p_n/q_n, and therefore by our result above, α is irrational.

Irrationality and Transcendence, VII

As first observed by Liouville, we can extend this criterion to exclude algebraic numbers that are roots of higher-degree polynomials by increasing the exponent of q in the bound on the right-hand side. First, some preliminaries:

- We say that a number $\alpha \in \mathbb{C}$ is algebraic if α is the root of some nonzero polynomial $p(x)$ with rational coefficients.
- If we consider all of the possible polynomials in $\mathbb{Q}[x]$ of which α is a root, by the well-ordering principle we can see that there is some polynomial of minimal degree d of which α is a root.
- We refer to this degree d as the algebraic degree of α over $\mathbb{Q}$. There is a unique monic polynomial of this degree d of which α is a root; this polynomial is called the minimal polynomial of α over $\mathbb{Q}$.

Irrationality and Transcendence, VIII

Examples:

1. Quadratic irrationals have algebraic degree 2 over $\mathbb{Q}$, since they are roots of quadratic polynomials but not any polynomial of lower degree.
2. The number $\sqrt[4]{2}$ has minimal polynomial $x^4 - 2$ over $\mathbb{Q}$ (although this is not completely trivial to prove) and thus has algebraic degree 4.
3. The number $\sqrt{2} + \sqrt{3}$ is algebraic because it is the root of the polynomial $x^4 - 5x^2 + 1$, and in fact (though this is again harder to prove) that polynomial is its minimal polynomial.
4. The number π is not algebraic because it is not the root of any nonzero polynomial with rational coefficients. (This is even harder to prove, of course.)

Irrationality and Transcendence, IX

Some other observations:

- The minimal polynomial is always irreducible (otherwise, one factor would have α as a root and have smaller degree) and it cannot have any repeated roots (otherwise m and its derivative m' would have a factor $x - \alpha$ in common).
- We can also clear denominators to see that any algebraic number is the root of a polynomial with integer coefficients.
- If α is the root of some polynomial $c_d x^d + c_{d-1} x^{d-1} + \cdots + c_0$ where the c_i are integers, then $c_d \alpha^d + c_{d-1} \alpha^{d-1} + \cdots + c_0 = 0$.
- If we set $\beta = c_d \alpha$, by rescaling we can see that β is a root of the polynomial $x^d + c_{d-1} c_d x^{d-1} + \cdots + c_0 c_d^{d-1}$, which is monic and has integer coefficients.
- Thus, up to an integer factor, any algebraic number is the root of a monic polynomial with integer coefficients.

Irrationality and Transcendence, X

With these preliminaries finished, we can now give Liouville's result:

Theorem (Liouville's Approximation Theorem)

Suppose α is algebraic of degree $n > 1$ over $\mathbb{Q}$ and that its minimal polynomial $m(x)$ has integer coefficients. Then there exists a positive real number A such that $|\alpha - p/q| \geq A/q^n$ for any rational number p/q.

The idea of the proof is to use the mean value theorem to bound the difference between $m(\alpha)$ and $m(p/q)$ and the fact that we can express $m(p/q)$ as $1/q^n$ times an integer.

We can also reduce to the situation where the minimal polynomial is monic by rescaling α, as we noted on the last slide. So the given assumption is not really a restriction.

Irrationality and Transcendence, XI

Proof:

- Suppose α is algebraic of degree $n > 1$ over $\mathbb{Q}$ and that its minimal polynomial $m(x)$ has integer coefficients and factors as $m(x) = (x - \alpha)(x - \beta_1)(x - \beta_2) \cdots (x - \beta_{n-1})$ over $\mathbb{C}$.
- Note that the β_i are distinct from α because $m(x)$ cannot have repeated roots as we noted earlier.
- Now define M be the maximum value of $|m'(x)|$ on the interval $[\alpha - 1, \alpha + 1]$, and set $A = \min(1, 1/M, |\alpha - \beta_i|)$ over all of the roots β_i.
- We claim this value of A satisfies the given inequality.
- To show this, suppose otherwise, so that p/q is rational and has $|\alpha - p/q| < A/q^n$. Then because $A \leq 1$, we have $p/q \in (\alpha - 1, \alpha + 1)$.
- Also, because $A \leq |\alpha - \beta_i|$, we see that $p/q \neq \beta_i$ for any i, and there is no root of $m(x)$ between α and p/q.

Irrationality and Transcendence, XII

Proof (continued):

- Now write $m(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_0$.
- Then $m(p/q) = (p/q)^d + c_{d-1}(p/q)^{d-1} + \cdots + c_0$ $= (1/q^d) \cdot [p^d + c_{d-1}p^{d-1}q + \cdots + c_0 q^d]$.
- So $|m(p/q)| \geq 1/q^d \cdot \left|p^d + c_{d-1}p^{d-1}q + \cdots + c_0 q^d\right| \geq 1/q^d$ because the term inside the absolute values is an integer and it cannot be zero since $m(p/q) \neq 0$.
- Now, by the mean value theorem, there exists x_0 in the interval with endpoints p/q and α such that $m(\alpha) - m(p/q) = m'(x_0) \cdot (\alpha - p/q)$. Taking absolute values yields $|m(\alpha) - m(p/q)| = |m'(x_0)| \cdot |\alpha - p/q|$.
- By assumption we have $A \leq 1/M$ and $|m'(x_0)| \leq M$, and also $m(\alpha) = 0$ and $|m(p/q)| \geq 1/q^d$.
- So we get $|\alpha - p/q| = \dfrac{|m(p/q)|}{|m'(x_0)|} \geq \dfrac{A}{q^d}$ as desired.

Irrationality and Transcendence, XIII

Roughly speaking, this result says that if we have an algebraic number α of degree d, then we cannot find a rational approximation that is "too close" to α.

- If we flip the condition around, then if we have a real number α that we *can* approximate extremely well, then it cannot be algebraic.
- More precisely: if α is an irrational real number such that there exists a constant $c > 0$ and a sequence of rational numbers p_n/q_n such that $|\alpha - p_n/q_n| < c/q_n^n$, then α is transcendental.
- The point is that this sequence of rational numbers contradicts the assertion that α is algebraic of degree n for every n, by Liouville's theorem, and so α must be transcendental.

Irrationality and Transcendence, XIV

We can use a similar sort of construction as we used earlier to construct transcendental numbers.

- We can construct such an α and corresponding rational approximations p_n/q_n by taking α to be an infinite series whose terms drop in size very quickly.
- Here, we want the tail after the nth partial sum p_n/q_n to be on the order of $1/q_n^n$ rather than $1/q_n^2$: this will guarantee that α will be transcendental.

Irrationality and Transcendence, XV

Example: Show that $\alpha = \sum_{k=0}^{\infty} 10^{-k!}$ is transcendental.

- Let $p_n/q_n = \sum_{k=0}^{n} 10^{-k!}$ be the nth partial sum of the series. We observe that $q_n = 10^{k!}$ since each of the other terms has a denominator dividing $10^{-k!}$.
- Furthermore, it is easy to see (e.g., from the decimal expansion of α) that the size of the tail $\sum_{k=n+1}^{\infty} 10^{-k!}$ is at most $2 \cdot 10^{-(n+1)!}$.
- Then we have an easy bound $|\alpha - p_n/q_n| < 2 \cdot 10^{-(n+1)!} = 2(10^{-n!})^{n+1} = 2/q_n^{n+1} < 1/q_n^n$. Since all of the partial sums of this series are distinct, we obtain infinitely many such p_n/q_n, and therefore by our result above, α is transcendental.

Summary

We finished our discussion of periodic continued fractions.

We discussed how to use continued fractions to do rational approximation.

We discussed some methods for constructing irrational and transcendental numbers.

Next lecture: Pell's Equation (part 1).