Math 4527 (Number Theory 2)

Lecture #5 of 38 $\sim$ January 28, 2021

Continued Fractions (Part 2)

- Computing Infinite Continued Fractions
- Periodic Continued Fractions

This material represents §6.2.3 from the course notes.

Reminders, I

Definition

A finite continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}},$$ *where the a_i are positive real numbers. For brevity, we will denote this expression using the much more compact notation $[a_0, a_1, \cdots, a_k]$. If the a_i are integers, we term it a simple continued fraction.*

Definition

If $C = [a_0, a_1, \cdots, a_k]$ is given, then the continued fraction $C_n = [a_0, a_1, \cdots, a_n]$ for $n < k$ is called the nth convergent to C.

Reminders, II

Here are some simple properties of continued fraction convergents:

Proposition (Properties of Convergents)

Let $C = [a_0, a_1, \ldots, a_k]$ where the a_i are positive, and define $p_{-1} = 1$, $p_0 = a_0$, and $p_n = a_n p_{n-1} + p_{n-2}$, and also $q_{-1} = 0$, $q_0 = 1$, and $q_n = a_n q_{n-1} + q_{n-2}$.

1. *The convergent $C_n = p_n/q_n$.*
2. *We have $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ and $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$.*
3. *We have $C_n - C_{n-1} = \dfrac{(-1)^{n-1}}{q_{n-1} q_n}$ and $C_n - C_{n-2} = \dfrac{(-1)^{n-2} a_n}{q_{n-2} q_n}$.*
4. *We have $C_1 > C_3 > C_5 > \cdots > C_6 > C_4 > C_2$, and $|C - C_n| \leq \dfrac{1}{q_n q_{n+1}} < \dfrac{1}{q_n^2}$.*

Reminders, III

Definition

Given a sequence a_0, a_1, a_2, ... of positive integers, we define the *<u>infinite continued fraction</u>* $\alpha = [a_0, a_1, a_2, \ldots]$ *to be the limit* $\lim_{n\to\infty} [a_0, a_1, \ldots, a_n]$ *of its finite continued fraction convergents.*

As we showed last time, every irrational real number has a unique infinite continued fraction expansion, which we can find numerically using our procedure of "subtract off the integer part, reciprocate, and repeat".

Many computer algebra systems have commands for computing continued fraction expansions. For example, in Mathematica, the command `ContinuedFraction[`α`,`n`]` will compute the first n terms of the expansion for α.

Continued Fractions Continued, I

Example: Find the continued fraction expansion of $8 + \sqrt{6}$.

Continued Fractions Continued, I

Example: Find the continued fraction expansion of $8 + \sqrt{6}$.

- With $\alpha = 8 + \sqrt{6}$, we find, successively,

n	0	1	2	3	4	$\cdots$
α_n	$8 + \sqrt{6}$	$\dfrac{\sqrt{6}+2}{2}$	$2 + \sqrt{6}$	$\dfrac{\sqrt{6}+2}{2}$	$2 + \sqrt{6}$	$\cdots$
a_n	10	2	4	2	4	$\cdots$
$\alpha_n - a_n$	$\sqrt{6} - 2$	$\dfrac{\sqrt{6}-2}{2}$	$\sqrt{6} - 2$	$\dfrac{\sqrt{6}-2}{2}$	$\sqrt{6} - 2$	$\cdots$

and since each term after this will repeat, we see that $8 + \sqrt{6} = \boxed{[10, 2, 4, 2, 4, 2, 4, \ldots]}$.

Continued Fractions Continued, VII

Example: Find the continued fraction expansion of π.

Continued Fractions Continued, VII

Example: Find the continued fraction expansion of π.

- It seems very unlikely we should expect to find a nice pattern for π, especially since we cannot even simplify any of the remainder terms (they will just be rational functions of π).
- The first 15 terms are $\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, \dots]$. No particular pattern seems apparent here.

Continued Fractions Continued, VIII

Example: Find the continued fraction expansion of $\sqrt{7}$.

Continued Fractions Continued, VIII

Example: Find the continued fraction expansion of $\sqrt{7}$.

- With $\alpha = \sqrt{7}$, we find, successively,

$$\begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & \cdots \\ \alpha_n & \sqrt{7} & \dfrac{\sqrt{7}+2}{3} & \dfrac{\sqrt{7}+1}{2} & \dfrac{\sqrt{7}+1}{3} & \sqrt{7}+2 & \cdots \\ a_n & 2 & 1 & 1 & 1 & 4 & \cdots \\ \alpha_n - a_n & \sqrt{7}-2 & \dfrac{\sqrt{7}-1}{3} & \dfrac{\sqrt{7}-1}{2} & \dfrac{\sqrt{7}-2}{3} & \sqrt{7}-2 & \cdots \end{array}$$

and since each term after this will repeat, we see that $\sqrt{7} = \boxed{[2, 1, 1, 1, 4, 1, 1, 1, 4, \dots]}$.

Continued Fractions Continued, IX

In some of the examples, the continued fraction expansion eventually started repeating. Let's now examine such expansions a bit more:

Definition

An infinite continued fraction $[a_0, a_1, a_2, \ldots]$ is (eventually) periodic if there is some integer n such that $a_r = a_{n+r}$ for all sufficiently large r. We employ the notation $[a_0, a_1, a_2, \ldots, a_k, \overline{a_{k+1}, a_{k+2}, \ldots, a_{k+n}}]$ to indicate that the block of integers under the bar repeats indefinitely.

This is the same notation used for repeating decimals.

Continued Fractions Continued, X

Example: Find the real number $\alpha = [\overline{1}]$ and its first ten convergents.

- By the periodicity of the expansion, we know that $\alpha = 1 + \dfrac{1}{\alpha} = \dfrac{\alpha+1}{\alpha}$.
- This yields a quadratic equation for α, namely $\alpha^2 = \alpha + 1$, whose solutions are $\alpha = \dfrac{1 \pm \sqrt{5}}{2}$. Since $\alpha > 1$, we need the plus sign, so $\alpha = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio.
- We can compute the convergents explicitly: the first ten are 1, 2, $\dfrac{3}{2}$, $\dfrac{5}{3}$, $\dfrac{8}{5}$, $\dfrac{13}{8}$, $\dfrac{21}{13}$, $\dfrac{34}{21}$, $\dfrac{55}{34}$, and $\dfrac{89}{55}$.

Continued Fractions Continued, XI

Notice that the convergents of $\alpha = [\overline{1}]$ are ratios of consecutive Fibonacci numbers.

- This is easy to show using the definition of α: by definition we have $\dfrac{p_{n+1}}{q_{n+1}} = 1 + \dfrac{1}{p_n/q_n} = \dfrac{q_n}{p_n + q_n}$.
- Thus, $p_{n+1} = q_n$ and so we can write $q_{n+1} = p_n + q_n = q_{n-1} + q_n$.
- Along with the initial conditions $p_1 = q_2 = q_1 = 1$, this is precisely the definition of the Fibonacci numbers.
- In fact, our results about the convergence of the convergents provide a proof that $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \dfrac{1+\sqrt{5}}{2}$.

Continued Fractions Continued, XII

Example: Find the real number $\alpha = [\overline{2, 5}]$.

Continued Fractions Continued, XII

Example: Find the real number $\alpha = [\overline{2, 5}]$.

- By the periodicity of the expansion, we know that $\alpha = 2 + \cfrac{1}{5 + \cfrac{1}{\alpha}} = 2 + \dfrac{\alpha}{5\alpha + 1} = \dfrac{11\alpha + 2}{5\alpha + 1}$.
- This yields a quadratic equation for α, namely $\alpha(5\alpha + 1) = 11\alpha + 2$, whose solutions are $\alpha = \dfrac{5 \pm \sqrt{35}}{5}$.
- Since $\alpha > 1$, we need the plus sign, so $\alpha = \dfrac{5 + \sqrt{35}}{5}$.

Continued Fractions Continued, XIII

So far, all of the periodic continued fractions we have seen have been solutions of a quadratic polynomial in $\mathbb{Q}[x]$. This is not an accident:

Theorem (Periodic Continued Fractions)

If α has a periodic continued fraction, then α is an irrational root of a quadratic polynomial with integer coefficients.

We call an irrational root of a quadratic polynomial with integer coefficients a quadratic irrational. The quadratic irrationals are the numbers of the form $\dfrac{p+\sqrt{D}}{q}$ where p and q are integers and D is a positive nonsquare integer.

Proof:

- Let $\alpha = [a_0, a_1, a_2, \cdots, a_k, \overline{a_{k+1}, a_{k+2}, \cdots, a_{k+n}}]$, and $\gamma = [\overline{a_{k+1}, a_{k+1}, \cdots, a_{k+n}}]$.
- Then by the periodicity of the expansion, we have $\gamma = [a_{k+1}, \cdots, a_{k+n}, \gamma]$.
- Expanding this out yields $\gamma = \dfrac{p_{n-1}\gamma + p_{n-2}}{q_{n-1}\gamma + q_{n-2}}$, which is a quadratic equation for γ.
- Since γ is irrational (being an infinite continued fraction), we conclude that $\gamma = \frac{b+\sqrt{c}}{d}$ for some integers b, c, and d.
- Then $\alpha = [a_0, a_1, a_2, \cdots, a_k, \gamma]$ is also a rational function in γ (and irrational), so clearing the denominator shows that $\alpha = \frac{e+\sqrt{f}}{g}$ for some integers e, f, and g, which is also a quadratic irrational.

Continued Fractions Continued, XV

The converse of this theorem is true as well, which is to say, every quadratic irrational has a periodic continued fraction expansion. But it will take a fair bit more work. We will use an approach motivated by the arithmetic of $\mathbb{Q}(\sqrt{D})$.

Definition

Let α be a quadratic irrational. The minimal polynomial $m(x)$ of α is the unique quadratic polynomial of which α is a root having the form $ax^2 + bx + c$ for relatively prime integers a, b, c where $a > 0$. We also define the discriminant of α to be the value $b^2 - 4ac \in \mathbb{N}$.

In other settings, the minimal polynomial is assumed to be monic and have rational coefficients. We take integer coefficients in our definition here because we want to work with properties that rely on having integer coefficients rather than rational coefficients.

Continued Fractions Continued, XVI

Examples:

1. The minimal polynomial of $\sqrt{2}$ is $x^2 - 2$, of discriminant 8.
2. The minimal polynomial of the golden ratio $(1 + \sqrt{5})/2$ is $x^2 - x - 1$, of discriminant 5.
3. The minimal polynomial of $(3 + \sqrt{13})/7$ is $49x^2 - 42x - 4$, of discriminant 2548.
4. The minimal polynomial of $\sqrt{6}/5 - 3$ is $25x^2 + 150x + 219$, of discriminant 600.

It is not hard to see that the minimal polynomial exists and is well-defined. The discriminant of a quadratic irrational is also always a positive integer, since $b^2 - 4ac$ is the term under the square root in the quadratic formula.

Continued Fractions Continued, XVII

A few more definitions:

Definition

Suppose α is a quadratic irrational.

If $\alpha = \dfrac{p+\sqrt{D}}{q}$, its conjugate is defined to be $\overline{\alpha} = \dfrac{p-\sqrt{D}}{q}$. The conjugate is the other root of the minimal polynomial of α.

We also say that α is reduced if $\alpha > 1$ and also $-1/\overline{\alpha} > 1$.

In general, the minimal polynomial of α will be $q^2(x-\alpha)(x-\overline{\alpha}) = q^2x^2 - 2pqx + (p^2 - D)$ up to scaling by a divisor of q (the coefficients need not be relatively prime, since $p^2 - D$ could have a factor in common with q).

Continued Fractions Continued, XVIII

Examples:

1. The conjugate of $\alpha = \sqrt{2}$ is $\overline{\alpha} = -\sqrt{2}$. Then α is not reduced since $-1/\overline{\alpha} = \sqrt{2}/2$ is not greater than 1.

2. The conjugate of $\beta = (1+\sqrt{5})/2$ is $\overline{\beta} = (1-\sqrt{5})/2$. Then β is reduced since both β and $-1/\overline{\beta} = \beta$ are greater than 1.

3. The conjugate of $\gamma = \sqrt{7} - 2$ is $\overline{\gamma} = -\sqrt{7} - 2$. Then γ is not reduced since γ is not greater than 1.

By first establishing some results about reduced quadratic irrationals, we can now prove that every quadratic irrational has a periodic continued fraction expansion.

Continued Fractions Continued, XIX

Theorem (Quadratic Irrationals and Continued Fractions)

Let α be a quadratic irrational with discriminant D and α_n be the nth remainder term from the continued fraction expansion of α.

1. *The remainder term α_n has discriminant D for all $n \geq 1$.*
2. *If α is a reduced quadratic irrational, then α_n is also reduced.*
3. *There are only finitely many reduced quadratic irrationals of discriminant D.*
4. *The remainder term α_n is reduced for sufficiently large n.*
5. *The continued fraction expansion of a real number α is periodic if and only if α is a quadratic irrational.*
6. *The continued fraction expansion of a real number α is purely periodic (i.e., is of the form $\alpha = [\overline{a_0, a_1, \ldots, a_n}]$) if and only if α is a reduced quadratic irrational.*

Continued Fractions Continued, XX

1. The remainder term α_n has discriminant D for all $n \geq 1$.

Proof:

- It is enough to show α_1 has discriminant D, since then the result for all α_n follows by a trivial induction.
- So suppose α has minimal polynomial $m(x) = ax^2 + bx + c$ and write $\lfloor \alpha \rfloor = a_0$.
- Since $\alpha = a_0 + 1/\alpha_1$ this means $a(a_0 + 1/\alpha_1)^2 + b(a_0 + 1/\alpha_1) + c = 0$, whence $a(a_0\alpha_1 + 1)^2 + b(a_0\alpha_1 + 1)\alpha_1 + c(\alpha_1)^2 = 0$; equivalently, $(aa_0^2 + ba_0 + c)\alpha_1^2 + (2aa_0 + b)\alpha_1 + a = 0$.
- Since a, b, c are relatively prime, so are $aa_0^2 + ba_0 + c$, $2aa_0 + b$, and a.
- Thus up to sign, the minimal polynomial of α_1 is $(aa_0^2 + ba_0 + c)x + (2aa_0 + b)x + a$, and so its discriminant is $(2aa_0 + b)^2 - 4a(aa_0^2 + ba_0 + c) = b^2 - 4ac = D$, as claimed.

Continued Fractions Continued, XXI

2. If α is a reduced quadratic irrational, then α_n is also reduced.

Proof:

- We show that if α is reduced then α_1 is reduced, and then apply a trivial induction.
- If α is reduced, then $\alpha_1 = \dfrac{1}{\alpha - \lfloor \alpha \rfloor} > 1$ since $0 < \alpha - \lfloor \alpha \rfloor < 1$.
- Also, $\overline{\alpha}_1 = \dfrac{1}{\overline{\alpha} - \lfloor \alpha \rfloor}$ is negative because $\overline{\alpha}$ is negative, and its absolute value is between 0 and 1 because $\lfloor \alpha \rfloor \geq 1$. Thus, $-1/\overline{\alpha}_1 > 1$ as required, and so α_1 is reduced.

Continued Fractions Continued, XXII

3. There are only finitely many reduced quadratic irrationals of discriminant D.

Proof:

- Suppose α is a reduced quadratic irrational of discriminant D and minimal polynomial $m(x) = ax^2 + bx + c$, where $b^2 - 4ac = D$ and $a > 0$.
- Since $\alpha = \dfrac{-b + \sqrt{D}}{2a}$ is reduced, we have $-1/\overline{\alpha} > 1$ and so $-1 < \overline{\alpha} < 0$.
- Thus $\alpha + \overline{\alpha} = -b/a$ is positive, so since $a > 0$ that means $b < 0$.
- Furthermore, $\overline{\alpha} = \dfrac{-b - \sqrt{D}}{2a}$ and $a > 0$, this requires $-b - \sqrt{D} < 0$ and so $b > -\sqrt{D}$. Thus $-\sqrt{D} < b < 0$ and so there are finitely many possible b.

Continued Fractions Continued, XXIII

3. There are only finitely many reduced quadratic irrationals of discriminant D.

Proof (continued):

- We just showed $-\sqrt{D} < b < 0$.
- Then since $\alpha = \dfrac{-b+\sqrt{D}}{2a}$ must have $\alpha > 1$, we see that $a < -b + \sqrt{D} < 2\sqrt{D}$.
- Since a is positive, there are finitely many possible a.
- Then, finally, since $c = (b^2 - D)/(4a)$, there are finitely many possible triples (a, b, c) and thus finitely many possible α.

Continued Fractions Continued, XXIV

4. The remainder term α_n is reduced for sufficiently large n.

Proof:

- By definition, for any $n \geq 1$, we have $\alpha_n = \frac{1}{\alpha_{n-1} - \lfloor \alpha_{n-1} \rfloor} > 1$. It remains to obtain a bound on $-1/\overline{\alpha_n}$.
- First, by definition we have $\alpha = [a_0, a_1, \ldots, a_n, \alpha_n]$, so if we set $[a_0, a_1, \ldots, a_n] = p_n/q_n$, then so that $\alpha = \dfrac{p_n \alpha_n + p_{n-1}}{q_n \alpha_n + q_{n-1}}$.
- Conjugating yields $\overline{\alpha} = \dfrac{p_n \overline{\alpha}_n + p_{n-1}}{q_n \overline{\alpha}_n + q_{n-1}}$ since the p_i and q_i are integers hence unchanged by conjugating.
- Rearranging this last expression gives $-\dfrac{1}{\overline{\alpha}_n} = -\dfrac{q_n \overline{\alpha} - p_n}{q_{n-1} \overline{\alpha} - p_{n-1}} = \dfrac{q_n}{q_{n-1}} \cdot \dfrac{\overline{\alpha} - p_n/q_n}{\overline{\alpha} - p_{n-1}/q_{n-1}}$.

Continued Fractions Continued, XXV

4. The remainder term α_n is reduced for sufficiently large n.

Proof (continued):

- We have $-\dfrac{1}{\overline{\alpha}_n} = -\dfrac{q_n\overline{\alpha} - p_n}{q_{n-1}\overline{\alpha} - p_{n-1}} = \dfrac{q_n}{q_{n-1}} \cdot \dfrac{\overline{\alpha} - p_n/q_n}{\overline{\alpha} - p_{n-1}/q_{n-1}}$.
- For large n, as we have shown, $p_n/q_n \to \alpha$, and thus the second term approaches $\dfrac{\overline{\alpha} - \alpha}{\overline{\alpha} - \alpha} = 1$ (note that the denominator is nonzero because α is irrational).
- The first term q_n/q_{n-1} is always greater than 1, and its limit cannot equal 1 because $q_n \geq q_{n-1} + q_{n-2}$, so dividing by q_{n-1} and taking the limit would give $1 \geq 1 + 1$, impossible.
- Therefore, for sufficiently large n, we see $-1/\overline{\alpha}_n > 1$, and so α_n is reduced.

Continued Fractions Continued, XXVI

5. The continued fraction expansion of a real number α is periodic if and only if α is a quadratic irrational.

Proof:

- We proved earlier that if α has a periodic continued fraction expansion, then α is a quadratic irrational.
- For the converse, suppose α is a quadratic irrational of discriminant D. Then by (1), every remainder term in the continued fraction expansion of α has discriminant D.
- By (4), the nth remainder term is reduced for sufficiently large n. But by (3), there are only finitely many such remainder terms, so by the pigeonhole principle there must be at least one repetition somewhere.
- But once a remainder term repeats, the rest of the expansion will be the same, and so the expansion is periodic, as claimed.

Continued Fractions Continued, XXVII

6. The continued fraction expansion of a real number α is purely periodic (i.e., is of the form $\alpha = [\overline{a_0, a_1, \dots, a_n}]$) if and only if α is a reduced quadratic irrational.

Proof:

- First suppose α has a purely periodic expansion.
- Then $\alpha = [a_0, a_1, \dots, a_{kn}, \alpha]$ for every positive integer k. Since by (4) the remainders are eventually all reduced, this means α must be reduced.
- Conversely, suppose α is reduced. By (5) we know that the continued fraction expansion is eventually periodic, say with $\alpha_{k+n} = \alpha_k$ for some k and n.
- We will show that $\alpha_{k+n-1} = \alpha_{k-1}$. Then by iterating this fact, this implies $\alpha_{j+n} = \alpha_j$ for all $j \geq 0$.
- Then we see immediately that α has a periodic continued fraction expansion, as $a_{j+n} = \lfloor \alpha_{j+n} \rfloor = \lfloor \alpha_j \rfloor = a_j$ for all $j \geq 0$.

Continued Fractions Continued, XXVIII

6. The continued fraction expansion of a real number α is purely periodic (i.e., is of the form $\alpha = [\overline{a_0, a_1, \dots, a_n}]$) if and only if α is a reduced quadratic irrational.

Proof (continued):

- It remains to show that if α is reduced and $\alpha_{k+n} = \alpha_k$ then $\alpha_{k+n-1} = \alpha_{k-1}$. First, both α_{k+n} and α_k are reduced by (2).
- By definition we have $\alpha_{k+n} = \frac{1}{\alpha_{k+n-1} - a_{k+n-1}}$ and $\alpha_n = \frac{1}{\alpha_{n-1} - a_{n-1}}$, so conjugating and inverting yields $-\dfrac{1}{\overline{\alpha}_{n+k}} = a_{k+n-1} - \overline{\alpha}_{k+n-1}$ and $-\dfrac{1}{\overline{\alpha}_n} = a_{n-1} - \overline{\alpha}_{n-1}$.
- Since both $\overline{\alpha}_{k+n-1}$ and $\overline{\alpha}_{n-1}$ are between -1 and 0, we see $a_{k+n-1} = \lfloor -\dfrac{1}{\overline{\alpha}_{n+k}} \rfloor = \lfloor -\dfrac{1}{\overline{\alpha}_n} \rfloor = a_{n-1}$, as claimed.

Continued Fractions Continued, XXIX

<u>Example</u>: Find the continued fraction expansion of $(3+\sqrt{13})/4$.

- Note $\alpha = (3+\sqrt{13})/4 > 1$ has $-1/\overline{\alpha} = 3+\sqrt{13} > 1$, so α is reduced. Per (6) above, its continued fraction expansion will be purely periodic.
- With $\alpha = (3+\sqrt{13})/4$, we find, successively,

n	0	1	2	3	4	5
α_n	$\frac{3+\sqrt{13}}{4}$	$\frac{1+\sqrt{13}}{3}$	$\frac{2+\sqrt{13}}{3}$	$\frac{1+\sqrt{13}}{4}$	$3+\sqrt{13}$	$\frac{3+\sqrt{13}}{4}$
a_n	1	1	1	1	6	
$\alpha_n - a_n$	$\frac{-1+\sqrt{13}}{4}$	$\frac{-2+\sqrt{13}}{3}$	$\frac{-1+\sqrt{13}}{3}$	$\frac{-3+\sqrt{13}}{4}$	$-3+\sqrt{13}$	

and we can see at this point each term will repeat. Therefore, the continued fraction expansion is $\boxed{[\overline{1,1,1,1,6}]}$, which is indeed periodic.

Summary

We continued our discussion of infinite continued fractions.

We discussed periodic continued fractions and proved that periodic continued fractions correspond to quadratic irrationals.

Next lecture: Rational approximation via continued fractions.