Math 4527 (Number Theory 2)

Lecture #4 of 38 $\sim$ January 27, 2021

Continued Fractions (Part 1)

- Finite Continued Fractions
- Infinite Continued Fractions

This material represents §6.2.2-§6.2.3 from the course notes.

Continued Fractions, I

We now discuss another method for generating rational approximations of a given real number α.

- If we want to give an approximation to α, it will be of the form $a_0 + x$ where $a_0 = \lfloor \alpha \rfloor$ is the greatest integer less than or equal to α and $0 \leq x < 1$.
- In such a situation, we have $1/x > 1$, so we could approximate $1/x$ as an integer $a_1 = \lfloor 1/x \rfloor$, yielding an approximation to α of the form $a_0 + \dfrac{1}{a_1}$.
- For example, if we wanted to approximate π, we would compute $\lfloor \pi \rfloor = 3$, and then note $x = \pi - 3 = 0.141592\ldots$ has $1/x \approx 7.06251\ldots$, and so we get the well-known approximation to π of $3 + 1/7 = 22/7$.

Continued Fractions, II

Alternatively, instead writing $\alpha = a_0 + x$ and stopping after approximating $x \approx 1/n$, we could instead approximate $1/x$ in the same way.

- Specifically, we can write $1/x = a_1 + y$ where $a_1 = \lfloor 1/x \rfloor$ and $0 \leq y < 1$.
- We can continue this procedure as long as each of the rounded-off values are not exact integers.
- For π, the next step would be noting that $y = 1/x - 7 \approx 0.06251$ has $1/y \approx 15.9966$, and so we get an approximation $1/x - 7 \approx 16$, which yields an approximation to π of $3 + 1/(7 + 1/16) = 355/113 \approx 3.14159292$, which is accurate to 6 decimal places.

Continued Fractions, III

We can continue this procedure to generate increasingly accurate rational approximations of α as "continued fractions":

Definition

A finite continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}},$$

where the a_i are positive real numbers. For brevity, we will denote this expression using the much more compact notation $[a_0, a_1, \cdots, a_k]$. If the a_i are integers, we term it a simple continued fraction.

Continued Fractions, IV

Examples:

1. $[2,3,4] = 2 + \cfrac{1}{3 + \cfrac{1}{4}} = \frac{30}{13}.$
2. $[1,1,1,1,1] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}} = \frac{8}{5}.$
3. $[1/2, 1/3, 1/4] = \frac{1}{2} + \cfrac{1}{\frac{1}{3} + \cfrac{1}{1/4}} = \frac{19}{26}.$

Continued Fractions, V

We note a few basic properties of continued fractions that follow immediately from the definition:

1. $[a_0, a_1, \ldots, a_k] = a_0 + \dfrac{1}{[a_1, \ldots, a_k]}$.
2. $[a_0, a_1, \ldots, a_k] = [a_0, a_1, \ldots, a_{k-1} + \dfrac{1}{a_k}]$.
3. Every finite simple continued fraction is a rational number: thus, no irrational number can be written as a finite simple continued fraction.
4. Conversely, any rational number a/b can be written as a simple continued fraction, as follows from an easy induction on b. Explicitly, if $b = 1$ then $a/b = [a]$, and then for $b > 1$ if we divide to write $b = qa + r$, we have $a/b = q + 1/(b/r)$, and b/r has a smaller denominator than a/b.

Continued Fractions, VI

In fact, per the last slide, we can see a nice connection between the Euclidean algorithm and continued fractions.

- Specifically, suppose a/b is in lowest terms, and apply the Euclidean algorithm to write
$$\begin{aligned} a &= q_1 b + r_1 \\ b &= q_2 r_1 + r_2 \\ &\vdots \\ r_{k-1} &= q_k r_k + 1 \\ r_k &= q_{k+1} \end{aligned}$$
where we know the last remainder is $1 = \gcd(p, q)$.
- Then $\dfrac{a}{b} = q_1 + \dfrac{1}{b/r_1} = q_1 + \dfrac{1}{q_2 + 1/(r_1/r_2)} = \cdots$
$= [q_1, q_2, \ldots, q_k, q_{k+1}]$.

Continued Fractions, VII

Indeed, from this Euclidean algorithm calculation, we can see that the simple continued fraction expansion of any rational number is essentially unique.

- Specifically, by the uniqueness of the Euclidean algorithm, all of the quotients are unique.
- Thus, up to the length of the expression, it is unique.
- It is not hard to see that the only way to alter the length of the expression is to write $[q_1, q_2, \cdots, q_k] = [q_1, q_2, \cdots, q_k - 1, 1]$.
- If we exclude the case where the final term is equal to 1, then every positive rational number can be written uniquely as a continued fraction.

Continued Fractions, VIII

Example: Find the continued fraction expansion of $18/7$.

Continued Fractions, VIII

Example: Find the continued fraction expansion of $18/7$.

- Applying the Euclidean algorithm yields

$$\begin{aligned} 17 &= 2 \cdot 7 + 4 \\ 7 &= 1 \cdot 4 + 3 \\ 4 &= 1 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 \end{aligned}$$

- Reading off the quotients then yields $18/7 = [2, 1, 1, 3]$.
- Of course, we could just do this explicitly by rounding down and peeling off the integer part at each stage:

$$\frac{18}{7} = 2 + \frac{4}{7} = 2 + \frac{1}{7/4} = 2 + \frac{1}{1 + \frac{1}{4/3}} = 2 + \frac{1}{1 + \frac{1}{1 + 1/3}}.$$

Continued Fractions, IX

If we truncate a continued fraction after some number of terms, we will obtain an approximation to the true value.

Definition

If $C = [a_0, a_1, \cdots, a_k]$ *is given, then the continued fraction* $C_n = [a_0, a_1, \cdots, a_n]$ *for* $n < k$ *is called the nth* *convergent* *to* C.

Example:

- For $\dfrac{256}{221} = [1, 6, 3, 5, 2] \approx 1.15837$, we see $[1] = 1$,
  $[1, 6] = \dfrac{7}{6} \approx 1.66667$,
  $[1, 6, 3] = \dfrac{22}{19} \approx 1.15789$,
  $[1, 6, 3, 5] = \dfrac{117}{101} \approx 1.15842$.

Continued Fractions, X

Here are some simple properties of continued fraction convergents:

Proposition (Properties of Convergents)

Let $C = [a_0, a_1, \ldots, a_k]$ where the a_i are positive, and define $p_{-1} = 1$, $p_0 = a_0$, and $p_n = a_n p_{n-1} + p_{n-2}$, and also $q_{-1} = 0$, $q_0 = 1$, and $q_n = a_n q_{n-1} + q_{n-2}$.

1. *The convergent $C_n = p_n / q_n$.*
2. *We have $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ and $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$.*
3. *We have $C_n - C_{n-1} = \dfrac{(-1)^{n-1}}{q_{n-1} q_n}$ and $C_n - C_{n-2} = \dfrac{(-1)^{n-2} a_n}{q_{n-2} q_n}$.*
4. *We have $C_1 > C_3 > C_5 > \cdots > C_6 > C_4 > C_2$, and $|C - C_n| \le \dfrac{1}{q_n q_{n+1}} < \dfrac{1}{q_n^2}$.*

Continued Fractions, XI

1. The convergent $C_n = p_n/q_n$.

Proof:

- Induct on n. The base cases $n = 1$ and $n = 2$ are trivial, since $[a_0] = a_0/1$ and $[a_0, a_1] = a_0 + 1/a_1 = (a_0 a_1 + 1)/a_1$.
- First, observe that

$$[a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}] = [a_0, a_1, \ldots, a_{m-1}, a_m + \frac{1}{a_{m+1}}].$$

- Now by definition, $[a_0, a_1, \ldots, a_{m-1}, x] = \dfrac{p_{m-1}x + p_{m-2}}{q_{m-1}x + q_{m-2}}$.
- Thus, setting $x = a_m + \dfrac{1}{a_{m+1}}$ and simplifying gives

$$[a_0, a_1, \ldots, a_{m-1}, a_m + \frac{1}{a_{m+1}}] = \frac{a_{m+1}p_m + p_{m-1}}{a_{m+1}q_m + q_{m-1}} = \frac{p_{m+1}}{q_{m+1}}.$$

Continued Fractions, XII

2. We have $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ and $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$.

<u>Proof</u>:

- For the first statement, by the recursion we can write

$$\begin{aligned} p_n q_{n-1} - p_{n-1} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(a_n q_{n-1} - q_{n-2}) \\ &= -(p_{n-1} q_{n-2} - p_{n-2} q_{n-2}) \end{aligned}$$

  so since $p_1 q_0 - p_0 q_1 = 1$, by a trivial induction we see that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.
- The second statement follows in the same way. (Exercise for you, if you want.)

Continued Fractions, XIII

3. We have $C_n - C_{n-1} = \dfrac{(-1)^{n-1}}{q_{n-1}q_n}$ and $C_n - C_{n-2} = \dfrac{(-1)^{n-2}a_n}{q_{n-2}q_n}$.

Proof:

- For the first, divide $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ and $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$ from (2) by $q_n q_{n-1}$.
- This yields $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{(-1)^{n-1}}{q_{n-1}q_n}$.
- The second one follows by dividing the other relation from (2) by $q_n q_{n-2}$ respectively.

Continued Fractions, XIV

4. We have $C_1 > C_3 > C_5 > \cdots > C_6 > C_4 > C_2$, and $|C - C_n| \le \dfrac{1}{q_n q_{n+1}} < \dfrac{1}{q_n^2}$.

Proof:

- From $C_n - C_{n-2} = \dfrac{(-1)^{n-2} a_n}{q_{n-2} q_n}$ in (3), we see that $C_n < C_{n-2}$ if n is odd, and $C_n > C_{n-2}$ if n is even.
- Hence, by a trivial induction, we see $C_1 > C_3 > C_5 > \cdots$ and $\cdots > C_6 > C_4 > C_2$.
- Furthermore, since $C_{2n+1} > C_{2n}$ for every n, we can combine the two chains of inequalities to obtain the third statement.
- For the last statement, the inequalities above imply that C is between C_n and C_{n+1} for every n, hence the triangle inequality implies $|C - C_n| \le |C_{n+1} - C_n| = \dfrac{1}{q_n q_{n+1}} < \dfrac{1}{q_n^2}$.

Continued Fractions, XV

Part (4) of the previous proposition gives us a more precise estimate for how good the approximation of a number α by its continued fraction convergents can be.

- Specifically, it says that the estimate $C_n = p_n/q_n$ is within an error of $1/q_n^2$, which is very close.
- If, for example, we used an arbitrary denominator D, then we could have an error as large as $1/(2D)$ for the best estimate by a rational with denominator D.
- The continued fraction convergent does quite a lot better than this (since the denominator exponent is 2 rather than 1).
- Do note, however, that it is the same order of magnitude as the estimate from Farey fractions we got earlier: if α is irrational, then there are infinitely many distinct rational numbers p/q such that $|\alpha - p/q| < 1/q^2$.

Continued Fractions, XVI

What we would like to do now is extend our discussion of continued fractions to cover irrational numbers α.

- Of course, as we already noted, irrational numbers do not have a finite continued fraction expansion: so what, for example, would it mean to ask for the continued fraction expansion of $\sqrt{2}$, or of π, or $\ln(2)$?
- To handle this, we simply extend our definition of continued fraction to an infinite continued fraction by taking a limit.

Definition

Given a sequence a_0, a_1, a_2, ... of positive integers, we define the *infinite continued fraction* $\alpha = [a_0, a_1, a_2, \dots]$ *to be the limit* $\lim_{n\to\infty} [a_0, a_1, \dots, a_n]$ *of its finite continued fraction convergents.*

Continued Fractions, XVII

It is not clear a priori that the limit $\lim_{n\to\infty} [a_0, a_1, \cdots, a_n]$, but in fact it always does.

- From our results, for $C_n = [a_0, a_1, \cdots, a_n]$, we have $C_1 > C_3 > C_5 > \cdots > C_6 > C_4 > C_2$.
- Thus, the sequence C_1, C_3, C_5, ... is monotone decreasing and bounded below (by C_2), hence it has a limit by the monotone convergence theorem[1].
- Similarly, the sequence C_2, C_4, C_6, ... is monotone increasing and bounded above (by C_1), hence it also has a limit by the monotone convergence theorem.
- These limits must be equal because $|C_n - C_{n+1}| < 1/q_n^2 \to 0$.

[1] Any monotone increasing sequence that is bounded above (i.e., any sequence $a_1 < a_2 < a_3 < \cdots$ such that all terms are less than some finite number M) has a limit. By negating, any monotone decreasing sequence bounded below also has a limit.

Continued Fractions, XVIII

Some other ways to see that the limit $\lim_{n\to\infty}[a_0, a_1, \cdots, a_n]$ exists:

- We could observe that the intervals $[C_{2n}, C_{2n-1}]$ form a set of nested closed intervals of lengths tending to zero.
- Then by the nested intervals theorem[2], their intersection is a single point C equal to the limit of the sequence C_i.
- A third way is to note $|C_j - C_k| \leq |C_j - C_{j+1}| \leq 1/q_j^2$ for all $k \geq j$, so the sequence $C_1, C_2, \ldots$ is Cauchy.

[2]The nested intervals theorem says that if I_1, I_2, I_3, ... is an infinite sequence of nested closed intervals (i.e., where $I_{n+1} \subseteq I_n$ for each n) that are bounded, then the intersection $\cap_{n=1}^{\infty} I_n$ is also a closed interval. Furthermore, if the lengths of the intervals tend to zero, then the intersection consists of a single point.

Continued Fractions, XIX

We can now establish some of the basic properties of infinite continued fractions.

Proposition (Properties of Infinite Continued Fractions)

Let $\alpha = [a_0, a_1, a_2, \dots]$ be an infinite simple continued fraction with nth convergent $C_n = [a_0, a_1, \cdots, a_n] = p_n/q_n$. Then the following hold:

1. *We have $|\alpha - C_n| \leq \dfrac{1}{q_n q_{n+1}} < \dfrac{1}{q_n^2}$.*
2. *Any infinite continued fraction α is irrational. Furthermore, any two different irrational numbers have different infinite continued fraction expansions.*

The first part is immediate from the finite case we did earlier, since α lies between C_n and C_{n+1}. The second part has actual content.

Continued Fractions, XX

2. Any infinite continued fraction α is irrational. Furthermore, any two different irrational numbers have different infinite continued fraction expansions.

Proof:

- For the first statement, suppose $\alpha = p/q$ were rational. By the proposition above, we know that $0 < \left|\frac{p}{q} - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$, meaning that $0 < |pq_n - p_n q| < \frac{q}{q_n}$.
- However, $\frac{q}{q_n}$ goes to zero as $n \to \infty$, since q is fixed but q_n is a strictly increasing sequence. This is impossible, since if $q_n > q$ the expression $|pq_n - p_n q|$ would be an integer between 0 and 1.

Continued Fractions, XXI

2. Any infinite continued fraction α is irrational. Furthermore, any two different irrational numbers have different infinite continued fraction expansions.

<u>Proof</u> (continued):

- For the second statement, first observe that $C_0 < \alpha < C_1$, meaning that $a_0 < \alpha < a_0 + \dfrac{1}{a_1}$, so we see that $\lfloor \alpha \rfloor = a_0$.
- Next, observe that $\alpha =$
$$\lim_{n\to\infty} [a_0, a_1, \ldots, a_n] = \lim_{n\to\infty} \left(a_0 + \frac{1}{[a_1, \ldots, a_n]} \right) = a_0 + \frac{1}{[a_1, a_2, \ldots]}.$$
- Now suppose $\beta = [b_0, b_1, \cdots]$ and $\beta = \alpha$. By taking floors, we see that $b_0 = a_0$.
- Then $[b_1, b_2, \ldots] = \dfrac{1}{\beta - b_0} = \dfrac{1}{\alpha - a_0} = [a_1, a_2, \ldots]$. Taking floors again shows $b_1 = a_1$. Repeating the argument yields $b_i = a_i$ for every i, so α and β are identical.

Continued Fractions, XXII

So far, we have discussed infinite continued fractions from the perspective of an explicit construction.

- However, of course, we would like to calculate the actual continued fraction expansions of some actual real numbers.
- The calculation in the proposition above shows how we can do this.
- So we will start in with some actual calculations next time.

Summary

We established some results on rational approximation using Farey sequences.

We introduced finite continued fractions and established some of the properties of their convergents.

We introduced infinite continued fractions. Next lecture: More

with continued fractions.