E. Dummit's Math 4527 ∼ Number Theory 2, Spring 2021 ∼ Homework 8, due Thu Mar 18th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Determine, with brief reasons, whether each subset S is an ideal of the given ring R:

   (a) $R = F[x]$, $S =$ the set of polynomials whose coefficient of x is zero.

   (b) $R = \mathbb{Z}/18\mathbb{Z}$, $S = \{0, 3, 6, 9, 12, 15\}$, the set of multiples of 3 in R.

   (c) $R = \mathbb{Z}/15\mathbb{Z}$, $S = \{0, 4, 8, 12\}$, the set of multiples of 4 in R.

   (d) $R = \mathbb{Z} \times \mathbb{Z}$, $S = \{(a, a) : a \in \mathbb{Z}\}$.

   (e) $R = \mathbb{Z} \times \mathbb{Z}$, $S = \{(0, a) : a \in \mathbb{Z}\}$.

   (f) $R = F[x]$, $S = F[x^2]$, the polynomials in which only even powers of x appear.

   (g) $R = F[x]$, $S =$ the set of polynomials whose coefficients sum to zero.

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2. Let $R = \mathbb{Z}[\sqrt{7}]$ and consider the ideals $I = (3)$ and $J = (3, 1 + \sqrt{7})$.

   (a) Show that R/I contains exactly 9 residue classes. [Hint: They are $p+q\sqrt{7}+I$ for $p, q \in \{0, 1, 2\}$. Explain why.]

   (b) Write down the multiplication table for R/I, and identify which elements are units and which elements are zero divisors. Is I a prime ideal? A maximal ideal?

   (c) Show that R/J contains exactly 3 residue classes and identify them. Is J a prime ideal? A maximal ideal?

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Let R be a commutative ring with 1.

   (a) Show that the union of a collection of ideals of R is not necessarily an ideal of R.

   (b) If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is an increasing chain of ideals of R, show that the union $\bigcup_i I_i$ is an ideal of R.

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4. Let R be a commutative ring with 1 and let I and J be ideals of R.

   (a) Show that $I + J = \{a + b : a \in I, b \in J\}$, the set of all sums of elements of I and J, is an ideal of R.

   (b) Show that $I + J$ is the smallest ideal of R that contains both I and J. Deduce that if $I = (a_1, \dots, a_n)$ and $J = (b_1, \dots, b_m)$ then $I + J = (a_1, \dots, a_n, b_1, \dots, b_m)$.

   (c) Let a and b be positive integers with gcd d. Show that $(a) + (b) = (d)$ in $\mathbb{Z}$.

   (d) Show that $IJ = \{a_1b_1 + \cdots + a_nb_n, : a_i \in I, b_i \in J\}$, the set of finite sums of products of an element of I with an element of J, is an ideal of R.

   (e) If $I = (a_1, \dots, a_n)$ and $J = (b_1, \dots, b_m)$, show that $IJ = (a_1b_1, a_1b_2, \dots, a_nb_1, a_1b_2, \dots, a_nb_m)$.

   (f) Show that IJ is an ideal contained in $I \cap J$, and give an example where $IJ \neq I \cap J$.

   (g) If $I + J = R$, show that $IJ = I \cap J$. [Hint: There exist $x \in I$ and $y \in J$ with $x + y = 1$.]

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5. Let R be a commutative ring with 1 and define the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for integers $0 \leq k \leq n$. Prove the binomial theorem in R: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for any $x, y \in R$ and any $n > 0$.

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6. Let R be a commutative ring with 1. We say $x \in R$ is nilpotent if $x^n = 0$ for some positive integer n.

(a) Find the nilpotent elements of $\mathbb{Z}/12\mathbb{Z}$.

(b) If m is a positive integer, show that a is nilpotent in $\mathbb{Z}/m\mathbb{Z}$ if and only if every prime divisor of m also divides a.

(c) Show that the set of nilpotent elements of R forms an ideal of R; this ideal is called the nilradical of R. [Hint: Use the binomial theorem to establish closure under subtraction.]

(d) If x is nilpotent, show that $1 + x$ is a unit. [Hint: What is $(1+x)(1-x+x^2-x^3+\cdots)$?]
