E. Dummit's Math 4527 ~ Number Theory 2, Spring 2021 ~ Homework 4, due Thu Feb 18th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Use the super magic box to factor each of the following integers:
   (a) 437.
   (b) 8137.

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2. Find all solutions to the equation $\frac{1}{a} + \frac{2}{b} = \frac{1}{10}$ in positive integers (a, b).

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3. Find all solutions to the Diophantine equation $y^2 = x^4 + 2x^3 + 2x^2 + 4$.

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4. Find all integers n for which $n^3 - 10n^2 + 20n + 17$ is the cube of an integer.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. The goal of this problem is to solve the equation $x^y = y^x$ in positive rational numbers. Assume $x, y > 0$.
   (a) Prove that any rational solution to the equation is either of the form $(x, y) = (s, s)$ or of the form $(x, y) = ((1 + 1/u)^u, (1 + 1/u)^{u+1})$ or $((1 + 1/u)^{u+1}, (1 + 1/u)^u)$ for some rational number $u > 0$. [Hint: If $y > x$, set $y = (1 + 1/u)x$.]
   (b) Let $m \geq 2$. Show that the difference between any two positive consecutive mth powers is greater than m.
   (c) With notation as in part (a), suppose $u = n/m$ in lowest terms. Show that $m + n$ and n must both be mth powers and deduce that $m = 1$. [Hint: Write out x in terms of m, n and use the fact that $m+n, m, n$ are relatively prime.]
   (d) Conclude that the rational solutions to $x^y = y^x$ are of the form $(x, y) = (s, s)$ for rational s along with $(x, y) = ((1 + 1/n)^n, (1 + 1/n)^{n+1})$ or $((1 + 1/n)^n, (1 + 1/n)^{n+1})$ for integers n.
   (e) Find all integral solutions to $x^y = y^x$.

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6. Prove that the sum of the first n positive integers is a perfect square for infinitely many values of n, and find the first five such n.

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7. Prove that there are no integral solutions to the equation $x^2 + y^2 = 3z^2$ other than $(0, 0, 0)$. [Hint: Use a descent argument modulo 3.]

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8. Prove that there are no integral solutions to the equation $y^9 = x^2 + 2020^{2021}$. [Hint: Work modulo 19.]

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9. The goal of this problem is to give two different approaches for solving the Diophantine equation $(x^2 - xy - y^2)^2 = 1$ in positive integers.

(a) Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Prove that $(x, y) = (F_{n+1}, F_n)$ satisfies $(x^2 - xy - y^2)^2 = 1$ for every $n \geq 1$.

(b) Suppose $(x, y) = (a, b)$ is a solution to $(x^2 - xy - y^2)^2 = 1$. Show that $a \geq b$, and that if $a > b$ then $(x, y) = (b, a - b)$ is also a solution to the system.

(c) Prove that every solution to $(x^2 - xy - y^2)^2 = 1$ is of the form $(a, b) = (F_{n+1}, F_n)$ for some $n \geq 1$. [Hint: If $x > y$ use (b) to construct a smaller solution. Conclude the result via a descent/induction argument.]

(d) Alternatively, suppose that (x, y) is a solution to $\left|x^2 - xy - y^2\right| = 1$. Show that $\left|\frac{x}{y} - \frac{1+\sqrt{5}}{2}\right| < \frac{1}{2y^2}$.

[Hint: Let $t = \frac{x}{y} - \frac{1}{2}$ and then show that $t > \frac{\sqrt{5}}{2} + \frac{1}{2y^2}$ and $t < \frac{\sqrt{5}}{2} - \frac{1}{2y^2}$ both yield contradictions.]

(e) Deduce again that every solution to the system is of the form $(a, b) = (F_{n+1}, F_n)$.
