E. Dummit's Math 4527 ~ Number Theory 2, Spring 2021 ~ Homework 3, due Thu Feb 11th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

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**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Find the rational number with denominator less than $N$ closest to each of the following real numbers $\alpha$:
   (a) $\alpha = \sqrt{13}$, $N = 100$.
   (b) $\alpha = \sqrt{2}$, $N = 100$.
   (c) $\alpha = e$, $N = 10000$.

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2. For each value of $D$, (i) find the continued fraction expansion for $\sqrt{D}$, (ii) find the fundamental unit in the ring $\mathbb{Z}[\sqrt{D}]$, (iii) determine whether the Pell's equation $x^2 - Dy^2 = -1$ has a solution and if so find the smallest one, and (iv) find the smallest two solutions to the Pell's equation $x^2 - Dy^2 = 1$:
   (a) $D = 19$.
   (b) $D = 22$.
   (c) $D = 130$.
   (d) $D = 61$.

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**Part II:** Solve the following problems. Justify all answers with rigorous, clear arguments.

3. Find the smallest positive integer $n$ such that for all integers $m$ with $0 < m < 2020$, there exists an integer $k$ with $\frac{m}{2020} < \frac{k}{n} < \frac{m+1}{2021}$. (Make sure to prove that your value is the smallest possible.)
   - Remark: This is a variation of problem B1 from the 1993 Putnam exam.

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4. Prove that the real number $\alpha = \sum_{k=1}^{\infty} \frac{1}{(k!)^{k!}} = 1 + \frac{1}{2^2} + \frac{1}{6^6} + \frac{1}{24^{24}} + \cdots$ is transcendental.

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5. The goal of this problem is to prove that if $\alpha$ is an arbitrary irrational number, then the maximum constant $C$ for which there necessarily exist infinitely many $p/q$ with $\left|\alpha - \frac{p}{q}\right| < \frac{1}{Cq^2}$ is at most $\sqrt{5}$. So let $C > \sqrt{5}$.
   (a) Let $\varphi = \frac{1+\sqrt{5}}{2} = [1, 1, 1, 1, \ldots]$ be the golden ratio and suppose that $\left|\varphi - \frac{p}{q}\right| < \frac{1}{Cq^2}$. Show that $\frac{p}{q} = \frac{F_{n+1}}{F_n}$ for some positive integer $n$, where $F_n$ is the $n$th Fibonacci number (defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for each $n \geq 1$).
   (b) Suppose $\alpha = [a_0, a_1, a_2, \ldots]$. Show that $\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})}$.
   (c) Show that $\left|\varphi - \frac{F_{n+1}}{F_n}\right| = \frac{1}{F_n^2(\varphi + F_{n-1}/F_n)}$ and that $\lim_{n\to\infty} [\varphi + F_{n-1}/F_n] = \sqrt{5}$.
   (d) Deduce that if $C > \sqrt{5}$, then there are only finitely many rational numbers $p/q$ such that $\left|\varphi - \frac{p}{q}\right| < \frac{1}{Cq^2}$.

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6. The goal of this problem is to prove that if $p$ is a prime congruent to 1 modulo 4, then there is always a solution to the negative Pell equation $x^2 - py^2 = -1$. As we showed, there exists a minimal solution $(x_1, y_1)$ to $x^2 - py^2 = 1$ where $x, y$ are positive and minimal.

   (a) Show that $x_1$ is odd, $y_1$ is even, and that $\gcd(x_1 + 1, x_1 - 1) = 2$.

   (b) Show either that $x_1 - 1 = 2ps^2$, $x_1 + 1 = 2t^2$ or that $x_1 - 1 = 2s^2$ and $x_1 + 1 = 2pt^2$ for some positive integers $s, t$. [Hint: Use $x_1^2 - 1 = py^2$ and $\gcd(x_1 + 1, x_1 - 1) = 2$.]

   (c) With notation as in (b), show that if $x_1 - 1 = 2ps^2$ and $x_1 + 1 = 2t^2$ then $t^2 - ps^2 = 1$, contradicting the minimality of $(x_1, y_1)$. Conclude in fact that there is an integer solution to $x^2 - py^2 = -1$.

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7. The goal of this problem is to establish some cases in which the negative Pell equation $x^2 - Dy^2 = -1$ has no solutions.

   (a) Suppose that $D$ is divisible by 4. Show that $x^2 - Dy^2 = -1$ has no solutions.

   (b) Suppose that $p$ is an odd prime and that there is a solution to the congruence $x^2 \equiv -1 \pmod{p}$. Prove that $p \equiv 1 \pmod{4}$. [Hint: Explain why $x$ has order 4 in the multiplicative group of nonzero residues modulo $p$, and then use Lagrange's theorem.]

   (c) Suppose that $D$ is divisible by a prime that is congruent to 3 modulo 4. Show that $x^2 - Dy^2 = -1$ has no solutions.

   (d) Observe that $x^2 - 2y^2 = -1$ and $x^2 - 17y^2 = -1$ both have integer solutions. It might stand to reason that $x^2 - 34y^2 = -1$ would also: show, however, that this equation does not have integer solutions.

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