E. Dummit's Math 4527 $\sim$ Number Theory 2, Spring 2021 $\sim$ Homework 2, due Thu Feb 4th.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. Compute the indefinite integral $\displaystyle\int \frac{1}{5+3\cos\theta}\,d\theta$. (Yes, you can just ask any computer to do this for you, but then you wouldn't learn how to do it yourself.)

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2. Express the following continued fractions as real numbers:
   (a) $[3, 1, 4, 1, 5]$.
   (b) $[\overline{1, 2, 3}]$.
   (c) $[\overline{3, 2, 1}]$.
   (d) $[\overline{3, 1, 2}]$.
   (e) $[3, \overline{1, 2}]$.

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3. Find the continued fraction expansions for the following:
   (a) $355/113$.
   (b) $418/2021$.
   (c) $13579/2468$.

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4. Find the continued fraction expansion, and the first five convergents, for each of the following:
   (a) $\sqrt{3}$.
   (b) $\sqrt{11}$.
   (c) $\dfrac{4+\sqrt{13}}{5}$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

5. The goal of this problem is to give a method for manipulating continued fractions using linear algebra. So suppose $a_0, a_1, \dots, a_n, \dots$ is a sequence of positive integers and set $p_n/q_n = [a_0, a_1, \dots, a_n]$ for each n.
   (a) Prove that $\begin{bmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$.
   (b) Show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$. [Hint: Determinant.]
   (c) Show that $[a_n, a_{n-1}, \dots, a_0] = p_n/p_{n-1}$ and that $[a_n, a_{n-1}, \dots, a_1] = q_n/q_{n-1}$. [Hint: Transpose.]

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6. Let α be the real number $\alpha = \sum_{n=1}^{\infty} \frac{n^2}{100^n} = 0.0104091625364964....$

   (a) Find the rational number p/q with smallest denominator that agrees with the expansion of α above, to the 16 decimal places shown.

   (b) Prove that your answer in part (a) is the only rational number with denominator less than 10^{10} whose decimal expansion agrees with that of α to the 16 decimal places shown.

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7. Let $\beta = [1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$ be the real number with continued fraction terms $a_{3i} = 1$, $a_{3i+1} = 2i$, and $a_{3i+2} = 1$ for each $i \geq 0$, and let $C_i = p_i/q_i$ be its convergents.

   (a) Show that the convergents $C_i = p_i/q_i$ have numerators and denominators satisfying the recurrences

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2} & \qquad q_{3n} &= q_{3n-1} + q_{3n-2} \\ p_{3n+1} &= 2np_{3n} + p_{3n-1} & \qquad q_{3n+1} &= 2nq_{3n} + q_{3n-1} \\ p_{3n+2} &= p_{3n+1} + p_{3n} & \qquad q_{3n+2} &= q_{3n+1} + q_{3n} \end{aligned}$$

   with initial values $p_0 = p_1 = q_1 = q_2 = 1$, $q_1 = 0$, and $p_2 = 2$.

   (b) Now define the integrals $A_n = \int_0^1 \frac{x^n(x-1)^n}{n!} e^x\, dx$, $B_n = \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x\, dx$, $C_n = \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x\, dx$. Show that $A_0 = e - 1$, $B_0 = 1$, and $C_0 = 2 - e$.

   (c) With notation as in part (b), show that $A_n = -B_{n-1} - C_{n-1}$, $B_n = -2nA_n + C_{n-1}$, and $C_n = B_n - A_n$. [Hint: For the first two, compute the derivatives of $\frac{1}{n!}x^n(x-1)^n e^x$ and $\frac{1}{n!}x^n(x-1)^{n+1}e^x$ and then integrate both sides.]

   (d) With notation as in part (c), show that $A_n = -(p_{3n} - q_{3n}e)$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$. [Hint: Note that A_n, B_n, C_n satisfy the same recurrences as p_n, q_n.]

   (e) Show that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = 0$.

   (f) Conclude that $\beta = \lim_{i\to\infty} p_i/q_i = e$, and from this fact deduce the continued fraction expansion of e.

   - Remark: This argument was originally given by Hermite, and is adapted from an expository article of H.A. Cohn.
