E. Dummit's Math 4527 ~ Number Theory 2, Spring 2021 ~ Homework 13, due Fri Apr 23rd.

Justify all responses with clear explanations and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers. Either staple the pages of your assignment together and write your name on the first page, or paperclip the pages and write your name on all pages.

Part I: No justifications are required for these problems. Answers will be graded on correctness.

1. For each quadratic integer ring (i) identify the value given by the Minkowski bound, (ii) find the splitting of all prime ideals up to the Minkowski bound, and (iii) determine the structure of the ideal class group:
   (a) $\mathbb{Z}[\sqrt{3}]$.
   (b) $\mathcal{O}_{\sqrt{13}}$.
   (c) $\mathbb{Z}[\sqrt{-6}]$.
   (d) $\mathbb{Z}[\sqrt{14}]$.
   (e) $\mathcal{O}_{\sqrt{-163}}$.
   (f) $\mathcal{O}_{\sqrt{-23}}$. [Hint: Show I_2^3 and $I_2 I_3$ are both principal.]

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2. Find reduced quadratic forms equivalent to $17x^2 - 83xy - 24y^2$, $16x^2 - 70xy + 77y^2$, and $77x^2 - 56xy + 10y^2$.

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3. For each discriminant Δ, find all reduced quadratic forms of discriminant Δ. For negative Δ, also compute the class number.
   (a) $\Delta = 12$.
   (b) $\Delta = 13$.
   (c) $\Delta = -24$.
   (d) $\Delta = -23$.
   (e) $\Delta = -163$.

Part II: Solve the following problems. Justify all answers with rigorous, clear arguments.

4. The goal of this problem is to give a geometric method for analyzing $\mathbb{Z}[i]/(\beta)$ for a nonzero Gaussian integer β.
   (a) Show that the ideal (β) forms a sublattice of the Gaussian integer lattice inside $\mathbb{C}$, and compute the area of its fundamental domain. [Hint: It is spanned by β and $i\beta$.]
   (b) Show that the number of residue classes in $\mathbb{Z}[i]/(\beta)$ is equal to the total number of interior points I, plus half of the number of boundary points B, minus one, inside the fundamental domain. [Hint: The boundary points come in pairs, except for the four corners.]
   (c) Deduce that the number of distinct residue classes in $\mathbb{Z}[i]$ modulo β is equal to $N(\beta)$. [Hint: Use Pick's theorem to put (a) and (b) together.]
   (d) Let $\beta = 3 + i$. Draw a fundamental region for $\mathbb{Z}[i]/(\beta)$, and use it to find an explicit list of residue class representatives for $\mathbb{Z}[i]/(\beta)$.
   (e) Does this method also work for $\mathcal{O}_{\sqrt{D}}/I$ for a general nonzero ideal I of $\mathcal{O}_{\sqrt{D}}$?

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5. Suppose $D < -2$ is a squarefree integer congruent to 2 or 3 modulo 4. As you showed on homework 11, the prime 2 ramifies in $\mathcal{O}_{\sqrt{D}}$, so that $(2) = P^2$ for a prime ideal P.
   (a) Show that P is not a principal ideal. [Hint: Consider the norm of a generator.]
   (b) Show that the class number of $\mathcal{O}_{\sqrt{D}}$ is even.

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6. [Optional] The goal of this problem is to prove the slightly sharper version of Minkowski's theorem for closed sets.
   (a) Suppose S is a closed subset of measure 1 inside $[0,1]^n$. Prove that $S = [0,1]^n$. [Hint: Consider the complement of S.]
   (b) Suppose S is a closed, bounded, measurable set in $\mathbb{R}^n$ whose n-measure is equal to 1. Show that there exist two points x and y in S such that $x - y$ has integer coordinates.
   (c) Suppose B be a convex closed set in $\mathbb{R}^n$ that is symmetric about the origin and whose n-measure is $\geq 2^n$. Prove that B contains a nonzero point all of whose coordinates are integers.

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7. Suppose $A = \{a_{i,j}\}_{1 \leq i,j \leq n}$ is a real $n \times n$ matrix whose determinant is not zero.
   (a) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive real numbers such that $\lambda_1 \lambda_2 \cdots \lambda_n \geq |\det A|$, prove that there exist integers $x_1, x_2, \ldots, x_n$, not all zero, such that $|a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n| \leq \lambda_1$, $|a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n| \leq \lambda_2$, ... , and $|a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n| \leq \lambda_n$.
   (b) Show that there exist nonzero integers a and b such that $\left|a\sqrt{3} + b\sqrt{22}\right|$ and $\left|a\sqrt{23} + 13b\right|$ are both less than $\dfrac{1}{\sqrt[4]{2024}}$.

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8. The goal of this problem is to prove a theorem of Euler about representations of integers by binary quadratic forms associated to $\mathcal{O}_{\sqrt{-5}}$.
   (a) If $p \neq 2, 5$ is a prime, show that p is represented by a binary quadratic form of discriminant $\Delta = -20$ if and only if $p \equiv 1, 3, 7, 9 \pmod{20}$.
   (b) Show that $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ are the only reduced positive-definite quadratic forms of discriminant $\Delta = -20$.
   (c) Show that $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ are not equivalent forms, and deduce that the class number of $\mathcal{O}_{\sqrt{-5}}$ is 2. [Hint: One form represents 2 while the other doesn't.]
   (d) Sharpen part (a) by showing that a prime $p \neq 2, 5$ is represented by $x^2 + 5y^2$ if and only if $p \equiv 1, 9 \pmod{20}$ and it is represented by $2x^2 + 2xy + 3y^2$ if and only if $p \equiv 3, 7 \pmod{20}$. [Hint: Show that p or $2p$ is a quadratic residue modulo 5, respectively.]
   (e) [Optional] Prove that an integer n can be represented in the form $x^2 + 5y^2$ or $2x^2 + 2xy + 3y^2$ if and only if each prime dividing n to an odd power is equal to 2 or 5 or is congruent to $1, 3, 7, 9$ modulo 20. Show furthermore that it is of the form $x^2 + 5y^2$ if and only if the total number of primes dividing n to an odd power that are congruent to 2, 3, or 7 modulo 20 is even.
   (f) Prove that if an integer n can be written in the form $x^2 + 5y^2$ for rational numbers x, y then it can be written in that form for integers x, y.

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9. Please provide some non-anonymous feedback on the course material:
   (a) If there were any topics I did not cover in the course that you think I should include the next time I teach it, please describe them.
   (b) If there were any topics I did cover in the course that you think I should spend less time on (or exclude entirely) the next time I teach it, please describe them.
   (c) Which topic was your favorite, and why? Which was your least favorite, and why?
   (d) Do you have any other comments about the course you would like to share?
