E. Dummit's Math 4527 ~ Number Theory 2, Spring 2021 ~ Homework 1, due Thu Jan 28th.

Justify all responses with proof and in complete sentences unless otherwise stated. Write up your solutions cleanly and neatly, and clearly identify all problem numbers.

---

**Part I:** No justifications are required for these problems. Answers will be graded on correctness.

1. Find all right triangles having one leg of length 40, and whose other two side lengths are integers.

---

2. Byzantine Basketball is like regular basketball except that foul shots are worth $a$ points instead of two points and field shots are worth $b$ points instead of three points. Moreover, in Byzantine Basketball there are exactly 35 scores that never occur in a game, one of which is 58. What are $a$ and $b$?

   **Remark:** This problem was on the 1971 Putnam exam.

---

3. Find all solutions in integers (if any) to the following linear Diophantine equations:
   (a) $22a + 17b = 19$.
   (b) $42a + 27b = 39$.
   (c) $3a + 7b + 16c = 8$.

---

4. Solve the following problems about Farey sequences:
   (a) Show that 7/13 and 13/24 are adjacent in the Farey sequence of level 24. What are the next three terms after them?
   (b) Find all $n$ such that exactly 2 terms appear between 7/13 and 13/24 in the Farey sequence of level $n$.
   (c) List all the terms between 6/19 and 5/14 in the Farey sequence of level 19.
   (d) Find the three terms following 154/227 in the Farey sequence of level 2020.
   (e) List all the terms between 1502/1801 and 1492/1789 in the Farey sequence of level 2021.
   (f) Find the least possible difference between two consecutive terms in the Farey sequence of level 2021.
   (g) Find the greatest possible difference between two consecutive terms in the Farey sequence of level 2021.

---

**Part II:** Solve the following problems. Justify all answers with rigorous, clear arguments.

5. If $n$ is any positive integer greater than 2, prove that there exists at least one right triangle with one side of length $n$ and whose other sides have integer lengths.

---

6. Prove that the area of any right triangle with integer sides is always divisible by 6, but not necessarily any integer greater than 6.

---

7. Find all solutions to the Diophantine equation $x^2 + y^2 = 2z^2$. [Hint: Any of the three methods used for $x^2 + y^2 = z^2$ can be used here. To factor in $\mathbb{Z}$, try setting $a = x + y$ and $b = x - y$; to factor in $\mathbb{Z}[i]$ use the factorization $2 = -i(1 + i)^2$; to use the geometry method, use lines through $(-1, 1)$.]

---

8. For each Farey fraction $a/b$, define $\mathcal{C}(a/b)$ to be the circle in the upper-half of the Cartesian plane of radius $r_{a/b} = 1/(2b^2)$ that is tangent to the $x$-axis at the point $(a/b, 0)$. These circles are called the Ford circles.
   (a) If $a/b$ and $c/d$ are adjacent entries in some Farey sequence, prove that $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are tangent.
   (b) If $a/b$ and $c/d$ are not adjacent in any Farey sequence, prove that the interiors of $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are disjoint.
   (c) Draw (you may use a computer) the 11 Ford circles corresponding to the Farey sequence of level 5.

---