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## 3 Multiple Integration

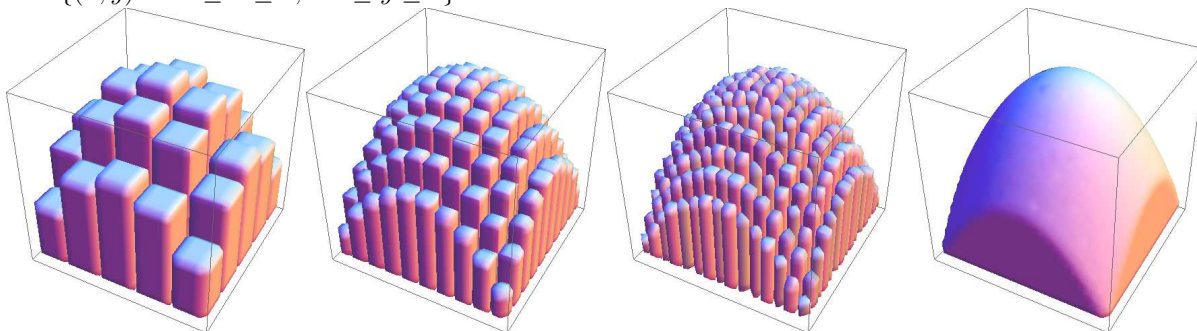
In this chapter we develop the theory of integration in multiple variables. Our focus is on double and triple integrals, as they are the ones that show up in most applications of multivariable calculus in a 3-dimensional universe.

We start by defining double integrals over regions in the plane (and triple integrals over a region in 3-space) in terms of Riemann sums, and then discuss how to evaluate double and triple integrals as “iterated integrals”. We then discuss how to set up double and triple integrals in alternative coordinate systems, focusing in particular on polar coordinates and their 3-dimensional analogues of cylindrical and spherical coordinates. We finish with some applications of multiple integration for finding areas, volumes, masses, and moments of solid objects.

### 3.1 Double Integrals in Rectangular Coordinates

- Our motivating problem for integration of one variable was to find the area below the curve  $y = f(x)$  above an interval on the  $x$ -axis. The motivating problem for double integrals is to find the volume below the surface  $z = f(x, y)$  above a region  $R$  in the  $xy$ -plane.
- Integrals in one variable are initially defined using Riemann sums, and we will do the same for double integrals.
  - The idea is the following: first, we approximate the region  $R$  by many small rectangular pieces, and in each piece we draw a rectangular prism with base in the  $xy$ -plane and upper face intersecting  $z = f(x, y)$ .

- Then we take the limit over better and better approximations of the region  $R$ , and (so we hope) the collective volume of the rectangular prisms will fill the volume under the graph of  $z = f(x, y)$ .
- Here is a series of such approximations for the volume under  $f(x, y) = 2 - x^2 - y^2$  on the region  $R = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ :



### 3.1.1 Double Integrals via Riemann Sums

- Here are the details of the formal definition of a double integral using Riemann sums:
- **Definition:** A region is a closed, bounded subset of the plane. If  $R$  is a region in the plane, a (tagged) partition of  $R$  into  $n$  pieces is a list of  $n$  disjoint rectangles inside  $R$ , where the  $k$ th rectangle contains the point  $(x_k, y_k)$ , has width  $\Delta x_k$ , height  $\Delta y_k$ , and area  $\Delta A_k = \Delta y_k \cdot \Delta x_k$ . The norm of the partition  $P$  is the largest number among the widths and heights of all of the rectangles in  $P$ .
- **Definition:** If  $P$  is a partition of the region  $R$ , and  $f(x, y)$  is a continuous function, we define the Riemann sum of  $f(x, y)$  on  $R$  corresponding to  $P$  to be  $RS_P(f) = \sum_{k=1}^n f(x_k, y_k) \Delta A_k$ .
  - **Reminder:** If  $g(x)$  is a function, then the notation  $\sum_{k=1}^n g(k)$  means  $g(1) + g(2) + g(3) + \dots + g(n)$ .
- **Definition:** For a continuous function  $f(x, y)$  defined on a region  $R$ , we define the (double) integral of  $f$  on  $R$ , denoted  $\iint_R f(x, y) dA$ , to be the value of  $L$  such that, for every  $\epsilon > 0$ , there exists a  $\delta > 0$  (depending on  $\epsilon$ ) such that for every partition  $P$  with  $\text{norm}(P) < \delta$ , we have  $|RS_P(f) - L| < \epsilon$ .
  - **Remark:** It can be proven (with significant effort) that, if  $f(x, y)$  is continuous, then a value of  $L$  satisfying the hypotheses actually does exist.
  - Essentially, what this definition means is: the value of the definite integral is the limit of the Riemann sums of  $f$ , as the size of the subregions in the partition becomes small.
  - Note that our geometric motivation for integration involved finding the area under the graph of a function  $z = f(x, y)$ . But as with a function of one variable, the definition via Riemann sums does not require that  $f$  be nonnegative, and we interpret the integral of a negative function as giving a negative volume.
- Like with integrals of a single variable, double integrals have a number of formal properties that can be deduced from the Riemann sum definition. Specifically, for  $C$  an arbitrary constant and  $f(x, y)$  and  $g(x, y)$  continuous functions, the following properties hold:
  - Integral of constant:  $\iint_R C dA = C \cdot \text{Area}(R)$ .
  - Constant multiple of a function:  $\iint_R C f(x, y) dA = C \cdot \iint_R f(x, y) dA$ .
  - Addition of functions:  $\iint_R f(x, y) dA + \iint_R g(x, y) dA = \iint_R [f(x, y) + g(x, y)] dA$ .
  - Subtraction of functions:  $\iint_R f(x, y) dA - \iint_R g(x, y) dA = \iint_R [f(x, y) - g(x, y)] dA$ .
  - Nonnegativity: if  $f(x, y) \geq 0$ , then  $\iint_R f(x, y) dA \geq 0$ .
    - \* As a corollary, if  $f(x, y) \geq g(x, y)$ , then by applying this property to  $f(x, y) - g(x, y) \geq 0$  and using the subtraction property, we see that  $\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$ .
  - Union: If  $R_1$  and  $R_2$  don't overlap and have union  $R$ , then  $\iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA = \iint_R f(x, y) dA$ .

### 3.1.2 Iterated Integrals and Fubini's Theorem

- Evaluating double integrals via Riemann sums is generally quite hard, even for very simple functions. However, we can give an alternative way, using the idea that volume is given by integrating cross-sectional area:
  - Suppose for simplicity that we want to find the volume underneath the portion of  $z = f(x, y)$  lying above the rectangular region  $a \leq x \leq b$ ,  $c \leq y \leq d$ .
  - If we imagine taking the solid volume and slicing it into thin pieces perpendicular to the  $x$ -axis from  $x = a$  to  $x = b$ , then the volume is given by the integral  $\int_a^b A(x) dx$ , where  $A(x)$  is the cross-sectional area at a given  $x$ -coordinate.
  - If we then look at each cross-section, we see that the area  $A(x_0)$  is simply the area under the curve  $z = f(x_0, y)$  between  $y = c$  and  $y = d$ , which is  $\int_c^d f(x_0, y) dy$ , where here we are thinking of  $x_0$  as a constant independent of  $y$ .
  - By putting all of this together, we see that the volume under  $z = f(x, y)$  on the region  $R$  should also be given by the iterated integral  $\int_a^b \left[ \int_c^d f(x, y) dy \right] dx$ , where we integrate first (on the inside) with respect to the variable  $y$ , and then second (on the outside) with respect to the variable  $x$ .
  - We will usually write iterated integrals without the brackets:  $\int_a^b \int_c^d f(x, y) dy dx$ .
  - Note that there are two limits of integration, and they are paired with the two differential variables “inside out”: the inner limits  $[c, d]$  are paired with the inner differential  $dy$ , and the outer limits  $[a, b]$  are paired with the outer differential  $dx$ .

- Example: Find the volume under the surface  $z = 6 - x^2 - y^2$  that lies above the region  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2$ .

- From the discussion above, we see that the volume is given by the iterated integral  $\int_0^1 \int_0^2 (6 - x^2 - y^2) dy dx$ .
- To evaluate the “inner” integral  $\int_0^2 (6 - x^2 - y^2) dy$ , we view  $x$  as a constant and take the antiderivative (with respect to  $y$ ):

$$\int_0^2 (6 - x^2 - y^2) dy = \left[ 6y - x^2y - \frac{1}{3}y^3 \right]_{y=0}^2 = \left( 12 - 2x^2 - \frac{8}{3} \right) - (0 - 0 - 0) = \frac{28}{3} - 2x^2.$$

- Now we can evaluate the “outer” integral  $\int_0^1 \left( \frac{28}{3} - 2x^2 \right) dx = \left[ \frac{28}{3}x - \frac{2}{3}x^3 \right]_{x=0}^1 = \boxed{\frac{26}{3}}$ .

- We can generalize this idea to compute the volume of a solid lying above a non-rectangular region.
  - Explicitly, suppose that the region  $R$  is defined by the inequalities  $a \leq x \leq b$ ,  $c(x) \leq y \leq d(x)$ : this represents the region above  $y = c(x)$  and below  $y = d(x)$ , between  $x = a$  and  $x = b$ .
  - Then by the same logic as before, the volume of the solid below  $z = f(x, y)$  above the region  $R$  in the  $xy$ -plane is given by the integral  $\int_a^b A(x) dx$ , where  $A(x)$  is the cross-sectional area at a given  $x$ -coordinate.
  - The area  $A(x_0)$  of each cross section will be the area under the curve  $z = f(x_0, y)$  between  $y = c(x_0)$  and  $y = d(x_0)$ , which is  $\int_{c(x_0)}^{d(x_0)} f(x_0, y) dy$ : here, again, we are thinking of  $x_0$  as a constant independent of  $y$ .
  - So, we see that the volume is given by the iterated integral  $\int_a^b \left[ \int_{c(x)}^{d(x)} f(x, y) dy \right] dx$ , where now the “inner limits” depend on  $x$ .

- Example: Find the volume under the surface  $z = 6 - x^2 - y^2$  that lies above the region  $0 \leq x \leq 1$ ,  $x \leq y \leq 2x$ .

- From the discussion above, we see that the volume is given by the iterated integral  $\int_0^1 \int_x^{2x} (6 - x^2 - y^2) dy dx$ .
- To evaluate  $\int_x^{2x} (6 - x^2 - y^2) dy$ , we take the antiderivative with respect to  $y$ :

$$\int_x^{2x} (6 - x^2 - y^2) dy = \left[ 6y - x^2y - \frac{1}{3}y^3 \right]_{y=x}^{2x} = \left( 12x - 2x^3 - \frac{8}{3}x^3 \right) - \left( 6x - x^3 - \frac{1}{3}x^3 \right) = 6x - \frac{10}{3}x^3.$$

◦ Now we can evaluate the “outer” integral  $\int_0^1 \left(6x - \frac{10}{3}x^3\right) dx = \left[3x^2 - \frac{5}{6}x^4\right]_{x=0}^1 = \boxed{\frac{13}{6}}$ .

- Notice also that instead of slicing perpendicular to the  $x$ -axis, we could have tried slicing perpendicular to the  $y$ -axis. By going through the same logic as before, the volume should be given by the iterated integral  $\int_c^d \int_a^b f(x, y) dx dy$  “in the other order”.
- Since we have three ways to interpret “the integral of a function on a region”, namely by Riemann sums and by iterated integration (in two possible orders), we would hope that these definitions all agree. It turns out that as long as the function is continuous on the entire region, they do:
- **Theorem** (Fubini): If  $f(x, y)$  is continuous on a region  $R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$  and  $R = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$ , then  $\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$ .

◦ Fubini’s Theorem says that in order to evaluate a double integral, we need only choose an order of integration, slice up the region accordingly, and then write down and evaluate the resulting iterated integral.

- Using Fubini’s Theorem, we can give the following process for evaluating double integrals:
  - **Step 0:** If necessary, determine the region of integration, and sketch it (if it is complicated).
  - **Step 1:** Decide on an order of integration and slice up the region according to the chosen order.
  - **Step 2:** Determine the limits of integration one at a time, starting with the outer variable. Note that the region may need to be split into several pieces, if the boundary of the region is complicated.
    - \* For  $R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ , the integral is  $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$ .
    - \* For  $R = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$ , the integral is  $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$ .
  - **Step 3:** Evaluate each iterated integral as a single-variable integral in the appropriate variable.
    - \* Remember that the outer variable is a constant, for the purposes of the inner integral. Think of the inner integration as “taking an anti-partial-derivative” and then evaluating at the endpoints, just like with a regular integral in one variable.

• **Example:** Evaluate  $I = \iint_R x^2 y dA$ , where  $R$  is the region  $\{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\}$ .

- We’ve been given the inequalities for the region, so we just need to write down the iterated integral. We can use either integration order  $dy dx$  or  $dx dy$ , so let’s do both.
- For  $dy dx$ , we have  $I = \int_0^2 \int_0^3 x^2 y dy dx$ . To evaluate the inner integral, we take the antiderivative of  $x^2 y$  with respect to  $y$ , yielding  $\frac{1}{2}x^2 y^2$ , and then plug in to evaluate the outer integral, as follows:

$$I = \int_0^2 \left[ \int_0^3 x^2 y dy \right] dx = \int_0^2 \left[ \frac{1}{2}x^2 y^2 \right]_{y=0}^3 dx = \int_0^2 \frac{9}{2}x^2 dx = \frac{3}{2}x^3 \Big|_{x=0}^2 = \boxed{12}.$$

- For  $dx dy$  we have  $I = \int_0^3 \int_0^2 x^2 y dx dy$ . To evaluate the inner integral, we take the antiderivative of  $x^2 y$  with respect to  $x$ , yielding  $\frac{1}{3}x^3 y$ , and then plug in like above:

$$I = \int_0^3 \left[ \int_0^2 x^2 y dx \right] dy = \int_0^3 \left[ \frac{1}{3}x^3 y \right]_{x=0}^2 dy = \int_0^3 \frac{8}{3}y dy = \frac{4}{3}y^2 \Big|_{y=0}^3 = \boxed{12}.$$

• **Example:** Evaluate  $\iint_R e^{x+y} dA$ , where  $R$  is the region  $\{(x, y) : 0 \leq x \leq \ln(2), 0 \leq y \leq x\}$ .

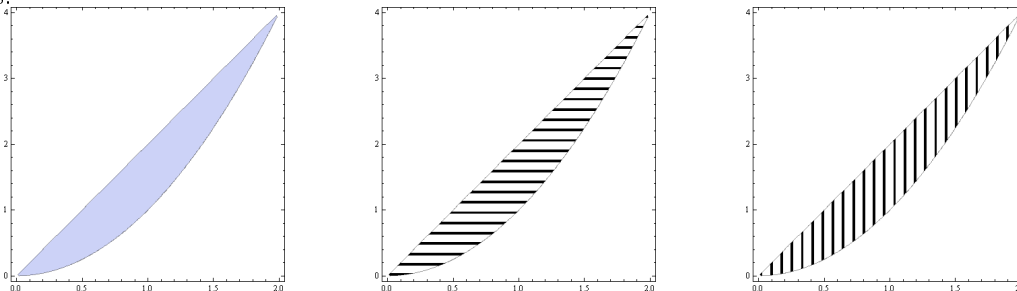
- Since the  $y$ -inequalities depend on  $x$ , the order  $dy dx$  is easiest: the resulting iterated integral is  $\int_0^{\ln(2)} \int_0^x e^{x+y} dA$ .

- Now we compute the inner integral, then plug in and evaluate the outer one:

$$\int_0^{\ln(2)} \left[ \int_0^x e^{x+y} dy \right] dx = \int_0^{\ln(2)} [e^{x+y}] \Big|_{y=0}^x dx = \int_0^{\ln(2)} [e^{2x} - e^x] dx = \left( \frac{1}{2}e^{2x} - e^x \right) \Big|_{x=0}^{\ln(2)} = \boxed{\frac{1}{2}}.$$

- Example:** Integrate the function  $f(x, y) = xy^2$  over the finite region  $R$  between the curves  $y = 2x$  and  $y = x^2$ .

- We can integrate in either order, so we will set up both integrals to illustrate the ideas in each case.
- We can see that the two curves  $y = 2x$  and  $y = x^2$  will intersect at  $(0, 0)$  and  $(2, 4)$ , so the region looks like this:



- We have two choices for setting up the integral: one with the order  $dx dy$  having horizontal slices, and the other with the order  $dy dx$  having vertical slices.
- For the order  $dy dx$ , we see that the allowable values for  $x$  range from  $0 \leq x \leq 2$ . Then for any given value of  $x$ , the allowable values of  $y$  range from the lower curve  $y = x^2$  to the upper curve  $y = 2x$ .

Therefore, the bounds on  $y$  are  $x^2 \leq y \leq 2x$ , and so the integral is  $\boxed{\int_0^2 \int_{x^2}^{2x} xy^2 dy dx}$ .

- We then compute

$$\int_0^2 \int_{x^2}^{2x} xy^2 dy dx = \int_0^2 \left[ \frac{1}{3}xy^3 \right] \Big|_{y=x^2}^{2x} dx = \int_0^2 \left[ \frac{8}{3}x^4 - \frac{1}{3}x^7 \right] dx = \left[ \frac{8}{15}x^5 - \frac{1}{24}x^8 \right] \Big|_{x=0}^2 = \boxed{\frac{8 \cdot 2^5}{15} - \frac{2^8}{24} = \frac{32}{5}}.$$

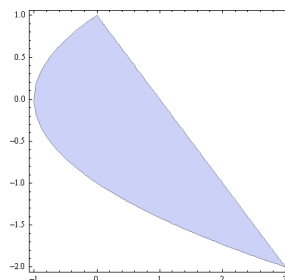
- For the order  $dx dy$ , we see that the allowable values for  $y$  range from  $0 \leq y \leq 4$ . Then for any given value of  $y$ , the allowable values of  $x$  range from the left curve  $y = 2x$  to the right curve  $y = x^2$ . Therefore, the bounds on  $x$  are  $\frac{1}{2}y \leq x \leq \sqrt{y}$ , and so the integral is  $\boxed{\int_0^4 \int_{y/2}^{\sqrt{y}} xy^2 dx dy}$ .

- We then compute

$$\int_0^4 \int_{y/2}^{\sqrt{y}} xy^2 dx dy = \int_0^4 \left[ \frac{1}{2}x^2y^2 \right] \Big|_{x=y/2}^{\sqrt{y}} dy = \int_0^4 \left[ \frac{1}{2}y^3 - \frac{1}{8}y^4 \right] dy = \left[ \frac{1}{8}y^4 - \frac{1}{40}y^5 \right] \Big|_{y=0}^4 = \boxed{\frac{4^4}{8} - \frac{4^5}{40} = \frac{32}{5}}.$$

- Example:** Integrate the function  $f(x, y) = y$  on the region  $R$  given by the finite area between the curves  $x = y^2 - 1$  and  $y = 1 - x$ .

- Here is a sketch of the region:



- To find the intersection points of the boundary curves  $y = 1 - x$  and  $x = y^2 - 1$ , plugging the first equation into the second yields  $x = (1 - x)^2 - 1 = x^2 - 2x$ , so that  $x = 0, 3$ . Therefore, the two intersection points are  $(0, 1)$  and  $(3, -2)$ . Also, the vertex of the parabola is  $(-1, 0)$ .

- We can set up the integral with either integration order, but to use vertical slices we need to divide the region into two pieces, since the top curve changes from the upper half of the parabola to the line at  $x = 0$ . If we use horizontal slices then it is not necessary to divide the region of integration.
- For horizontal slices, with integration order  $dx dy$ :

- \* We see that the range for  $y$  is  $-2 \leq y \leq 1$ , and then the corresponding range for  $x$  is  $y^2 - 1 \leq x \leq 1 - y$ .
- \* The desired integral is therefore  $\int_{-2}^1 \int_{y^2-1}^{1-y} y dx dy$ . Now we can evaluate it:

$$\begin{aligned} \int_{-2}^1 \int_{y^2-1}^{1-y} y dx dy &= \int_{-2}^1 [xy] \Big|_{x=y^2-1}^{1-y} dy = \int_{-2}^1 [y(1-y) - y(y^2-1)] dy \\ &= \int_{-2}^1 [2y - y^2 - y^3] dy = \left[ y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4 \right] \Big|_{y=-2}^1 = \boxed{-\frac{9}{4}} \end{aligned}$$

- For vertical slices, with integration order  $dy dx$ :
- \* The range for  $x$  is  $-1 \leq x \leq 3$ . We need to divide it into two pieces, since the upper curve changes at  $x = 0$ .
- \* Solving  $x = y^2 - 1$  for  $y$  in terms of  $x$  yields  $y = \sqrt{x+1}$  (the upper half of the parabola),  $y = -\sqrt{x+1}$  (the lower half of the parabola). The line segment is  $y = 1 - x$ .
- \* For  $-1 \leq x \leq 0$  the lower curve is the bottom half of the parabola  $y = -\sqrt{x+1}$  and the upper curve is the top half of the parabola  $y = \sqrt{x+1}$ , so the range for  $y$  is  $-\sqrt{x+1} \leq y \leq \sqrt{x+1}$ , and the integral is

$$\int_{-1}^0 \int_{-\sqrt{x+1}}^{\sqrt{x+1}} y dy dx = \int_{-1}^0 \left[ \frac{1}{2}y^2 \right] \Big|_{y=-\sqrt{x+1}}^{\sqrt{x+1}} dx = \int_{-1}^0 0 dx = 0.$$

- \* For  $0 \leq x \leq 3$  the lower curve is still the bottom half of the parabola  $y = -\sqrt{x+1}$  but the upper curve is now the line segment  $y = 1 - x$ , so the range for  $y$  is  $-\sqrt{x+1} \leq y \leq 1 - x$  and the integral is

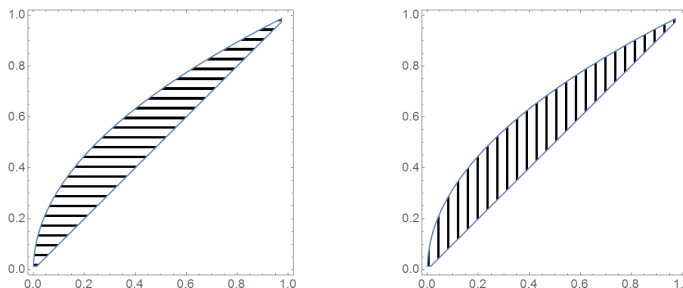
$$\begin{aligned} \int_0^3 \int_{-\sqrt{x+1}}^{1-x} y dy dx &= \int_0^3 \left[ \frac{1}{2}y^2 \right] \Big|_{y=-\sqrt{x+1}}^{1-x} dx = \int_0^3 \frac{1}{2} [(1-x)^2 - (x+1)] dx \\ &= \int_0^3 \frac{1}{2} [x^2 - 3x] dx = \frac{1}{2} \left[ \frac{1}{3}x^3 - \frac{3}{2}x^2 \right] \Big|_{x=0}^3 = -\frac{9}{4}. \end{aligned}$$

- \* The answer is then the sum  $\int_{-1}^0 \int_{-\sqrt{x+1}}^{\sqrt{x+1}} y dy dx + \int_0^3 \int_{-\sqrt{x+1}}^{1-x} y dy dx = \boxed{-\frac{9}{4}}$ .

### 3.1.3 Changing the Order of Integration

- Although Fubini's Theorem guarantees that integrating in either order will yield the same result, it can happen that one order of integration is easier to evaluate than the other. In such situations, reversing the order of integration can be useful.
- To change the order of integration in a multiple integral, follow these steps:
  - Step 1: Sketch the region of integration.
  - Step 2: Slice up the region according to the new order, and determine the new limits of integration one at a time, starting with the outer variable. Note that the region may need to be split into several pieces, if the boundary of the region is complicated.
  - Step 3: Evaluate the new integral.
- Example: Reverse the order of integration for  $\int_0^1 \int_{y^2}^y xy dx dy$ .

- The region is defined by  $0 \leq y \leq 1$  and  $y^2 \leq x \leq y$ , and the current order of integration has horizontal slices:

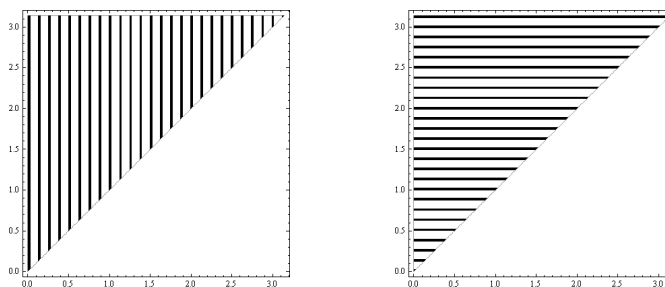


- To reverse the order of integration, we want to slice up the region perpendicular to the  $x$ -axis so that it has vertical slices, as in the second diagram above.
- We see that the range for  $x$  is  $0 \leq x \leq 1$ , and then the new limits for  $y$  are  $x \leq y \leq \sqrt{x}$ .

- Hence the integral becomes 
$$\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx.$$

- Example: Evaluate the integral  $\int_0^\pi \int_x^\pi \frac{\sin(y)}{y} \, dy \, dx$  by reversing the order of integration.

- As written it is not possible to evaluate the inner integral, as  $\frac{\sin(y)}{y}$  does not have an elementary antiderivative. So we try reversing the order of integration.
- The region is defined by  $0 \leq x \leq \pi$  and  $x \leq y \leq \pi$ , and the current order of integration has vertical slices:



- To reverse the order of integration, we want to slice up the region perpendicular to the  $y$ -axis so that it has horizontal slices, as in the second diagram above.
- We see that the range for  $y$  is  $0 \leq y \leq \pi$ , and then the new limits for  $x$  are  $0 \leq x \leq y$ .

- Then the integral is  $\int_0^\pi \int_0^y \frac{\sin(y)}{y} \, dx \, dy = \int_0^\pi \sin(y) \, dy = \boxed{2}.$

### 3.2 Triple Integrals in Rectangular Coordinates

- Now that we have discussed double integrals and integrating a function over a region in the plane, it is not hard to see how to generalize to triple integrals and integrating a function over a region in 3-space: we just increase the number of variables by 1, and integrate functions  $f(x, y, z)$  over regions in 3-space instead of functions  $f(x, y)$  over regions in the plane.
  - Definition: A region in 3-dimensional space is a closed, bounded subset of 3-space.
  - For clarity, we will use  $D$  to denote solid regions in 3-space, and reserve  $R$  for regions in the plane.
- The motivating problem for integration in three variables is somewhat less clear: for single integrals we wanted to find the area under a curve  $y = f(x)$ , and for double integrals we wanted to find the volume under a surface  $z = f(x, y)$ .

- For triple integrals it is somewhat harder to envision what happens when we move up by 1 dimension: we are then “finding the 4-dimensional volume under a 3-dimensional hypersurface” (whatever that means!).
- One way to interpret what a triple integral represents is to think of a function  $f(x, y, z)$  as being the density of a solid object  $D$  at a given point  $(x, y, z)$ : then the triple integral of  $f(x, y, z)$  on the region  $D$  represents the total mass of the solid.

### 3.2.1 Triple Integrals via Riemann Sums

- The definition of a triple integral via Riemann sums is essentially same as with double integrals: we approximate the region  $D$  by many small rectangular pieces, sum the function over all of the pieces, and then take the limit as the size of the pieces gets smaller.
  - Definition: For a region  $D$  a (tagged) partition of  $D$  into  $n$  pieces is a list of disjoint rectangular boxes inside  $D$ , where the  $k$ th rectangle contains the point  $(x_k, y_k, z_k)$ , has length  $\Delta x_k$ , width  $\Delta y_k$ , height  $\Delta z_k$ , and volume  $\Delta V_k = \Delta z_k \cdot \Delta y_k \cdot \Delta x_k$ . The norm of the partition  $P$  is the largest number among the dimensions of all of the boxes in  $P$ .
  - Definition: For  $f(x, y, z)$  a continuous function and  $P$  a partition of the region  $D$ , we define the Riemann sum of  $f(x, y, z)$  on  $D$  corresponding to  $P$  to be  $RS_P(f) = \sum_{k=1}^n f(x_k, y_k, z_k) \Delta V_k$ .
  - Definition: For  $f(x, y, z)$  a continuous function, we define  $\iiint_D f(x, y, z) dV$ , “the (triple) integral of  $f$  on the region  $D$ ”, to be the value of  $L$  such that, for every  $\epsilon > 0$ , there exists a  $\delta > 0$  (depending on  $\epsilon$ ) such that for every partition  $P$  with  $\text{norm}(P) < \delta$ , we have  $|RS_P(f) - L| < \epsilon$ .
  - Remark: It can be proven (with significant effort) that, if  $f(x, y, z)$  is continuous, then a value of  $L$  satisfying the hypotheses actually does exist.
- Triple integrals share the same formal properties as double integrals. Specifically, for  $C$  an arbitrary constant and  $f(x, y, z)$  and  $g(x, y, z)$  continuous functions, the following properties hold:
  - Integral of constant:  $\iiint_D C dV = C \cdot \text{Volume}(D)$ .
  - Constant multiple of a function:  $\iiint_D C f(x, y, z) dV = C \cdot \iiint_D f(x, y, z) dV$ .
  - Addition of functions:  $\iiint_D f(x, y, z) dV + \iiint_D g(x, y, z) dV = \iiint_D [f(x, y, z) + g(x, y, z)] dV$ .
  - Subtraction of functions:  $\iiint_D f(x, y, z) dV - \iiint_D g(x, y, z) dV = \iiint_D [f(x, y, z) - g(x, y, z)] dV$ .
  - Nonnegativity: if  $f(x, y, z) \geq 0$ , then  $\iiint_D f(x, y, z) dV \geq 0$ .
  - Union: If  $D_1$  and  $D_2$  don't overlap and have union  $D$ ,  $\iiint_{D_1} f(x, y, z) dV + \iiint_{D_2} f(x, y, z) dV = \iiint_D f(x, y, z) dV$ .

### 3.2.2 Fubini's Theorem and Methods of Evaluation

- Like with double integrals, we generally want to write triple integrals as iterated integrals, and avoid using Riemann sums. Once we have the integrals, the procedure for evaluating them is exactly the same:

- Example: Evaluate the integral  $\int_0^1 \int_0^y \int_x^y xyz dz dx dy$ .

- We just work one step at a time:

$$\begin{aligned}
 \int_0^1 \int_0^y \int_x^y xyz dz dx dy &= \int_0^1 \int_0^y \left[ \frac{1}{2}xyz^2 \right]_{z=x}^y dx dy \\
 &= \int_0^1 \int_0^y \left( \frac{1}{2}xy^3 - \frac{1}{2}x^3y \right) dx dy \\
 &= \int_0^1 \left( \frac{1}{4}x^2y^3 - \frac{1}{8}x^4y \right) \Big|_{x=0}^y dy \\
 &= \int_0^1 \frac{1}{8}y^5 dy = \boxed{\frac{1}{48}}.
 \end{aligned}$$

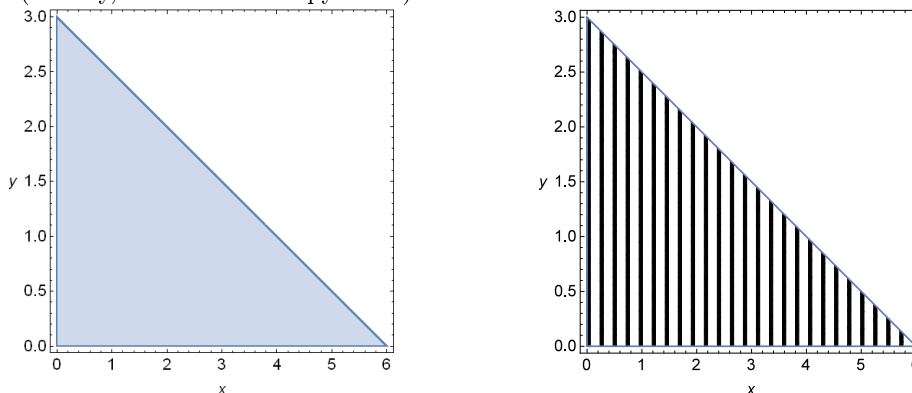


- Much of the difficulty in setting up triple integrals is converting the description of the region into explicit bounds of integration. To do this, we choose an order of integration, and then slice up the region of integration accordingly; however, things are complicated by the fact that we now have 3 variables instead of 2.
- We might also worry that the value of a triple integral might depend on the order of integration, but as we would hope, there is a version of Fubini's Theorem that guarantees the order of integration does not matter as long as the function is continuous:
- **Theorem** (Fubini): If  $f(x, y, z)$  is continuous on  $D = \{(x, y, z) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y)\}$ , then  $\iiint_D f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx$ , and all other orders of integration will also yield the same value.
- Here is the general process for setting up and evaluating triple integrals:
  - **Step 1:** If necessary, determine the region of integration, and sketch it.
  - **Step 2:** Decide on an order of integration and determine the limits of integration. Note that the region may need to be split into several pieces, depending on the order chosen.
    - \* The simplest method is to project the solid region into the plane spanned by the outer and middle variables, obtaining a region in that plane: then set up the outer and middle limits in the same way as for a double integral on that planar region.
    - \* Specifically: with the integration orders  $dz dy dx$  or  $dz dx dy$  we project into the  $xy$ -plane, with the orders  $dy dz dx$  or  $dy dx dz$  we project into the  $xz$ -plane, and with  $dx dz dy$  or  $dx dy dz$  we project into the  $yz$ -plane.
    - \* Then to find the bounds on the inner limit, we imagine moving parallel to the direction of the inner variable until we enter the region (the surface we pass through giving the lower bound of integration) and continuing until we leave the region (the surface we pass through giving the upper bound of integration).
    - \* Note that the bounds for the outer variable must be constants, the bounds for the middle variable can only depend on the outer variable, and the bounds for the inner variable can depend on both of the others.
  - **Step 3:** Evaluate each iterated integral as a single-variable integral in the appropriate variable.
    - \* Remember that all variables except the current variable of integration are to be treated as constants.
- **Example:** Find  $\iiint_D x dV$  where  $D$  is the tetrahedron bounded by the planes  $x = 0$ ,  $y = 0$ ,  $z = 0$ , and  $x + 2y + 3z = 6$ .
  - First, we sketch the region. It is a tetrahedron (triangular pyramid) whose vertices are  $(0, 0, 0)$ ,  $(6, 0, 0)$ ,  $(0, 3, 0)$ , and  $(0, 0, 2)$ :



- There are six possible orders of integration, and we could use any of the six. We will set up the integral in the order  $dz dy dx$ , which requires us to project this solid into the  $xy$ -plane.

- Motivated by the picture, we can see that the projection of the solid yields the interior of a triangle in the  $xy$ -plane (namely, the base of the pyramid):

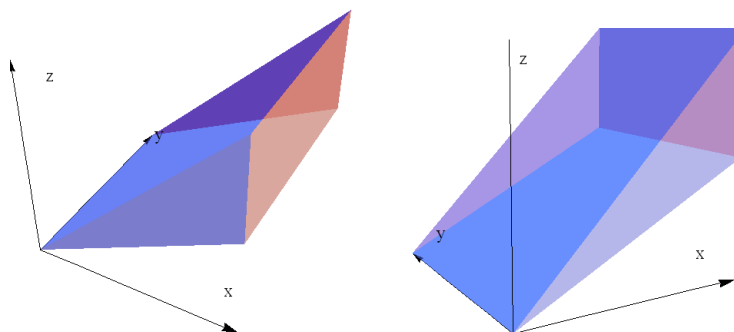


- We then cut up plane region with vertical slices (corresponding to the order  $dy dx$ ). Since the diagonal line has equation  $x + 2y = 6$ , which is the same as  $y = (6 - x)/2$ , we see that the region is described by  $0 \leq x \leq 6$  and  $0 \leq y \leq (6 - x)/2$ .
- For the bounds on  $z$ , we need to use the 3-dimensional picture of the solid. For fixed  $x$  and  $y$ , as we move in the direction of increasing  $z$ , we enter the solid through the  $xy$ -plane  $z = 0$  and exit the solid through the tilted plane  $z = (6 - x - 2y)/3$ , so the bounds on  $z$  are  $0 \leq z \leq (6 - x - 2y)/3$ .
- Hence the desired integral is  $\int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} x dz dy dx$ . Now we can evaluate it:

$$\begin{aligned}
 \int_0^6 \int_0^{(6-x)/2} \int_0^{(6-x-2y)/3} x dz dy dx &= \int_0^6 \int_0^{(6-x)/2} [xz]_{z=0}^{(6-x-2y)/3} dy dx \\
 &= \int_0^6 \int_0^{(6-x)/2} \left[ 2x - \frac{1}{3}x^2 - \frac{2}{3}xy \right] dy dx \\
 &= \int_0^6 \left[ 2xy - \frac{1}{3}x^2y - \frac{1}{3}xy^2 \right]_{y=0}^{(6-x)/2} dx \\
 &= \int_0^6 \left[ x(6-x) - \frac{1}{6}x^2(6-x) - \frac{1}{12}x(6-x)^2 \right] dx \\
 &= \int_0^6 \left[ 3x - x^2 + \frac{1}{12}x^3 \right] dx = \left[ \frac{3}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{48}x^4 \right]_{x=0}^6 = \boxed{9}.
 \end{aligned}$$

- Example: Find  $\iiint_D x^2 dV$ , where  $D$  is the “triangular wedge” bounded by the planes  $x = 0$ ,  $x = 1$ ,  $z = x$ ,  $z = 2x$ ,  $y = 0$ , and  $y = 1$ .

- Upon sketching the region, we see that it has a “prism” shape:

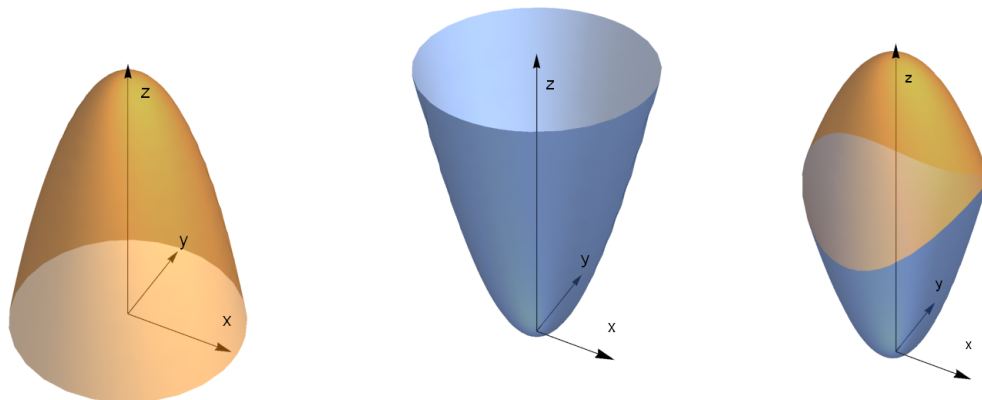


- Aided by the picture, it is easy to describe the projection of the region into the  $xz$ -plane as  $0 \leq x \leq 1$  and  $x \leq z \leq 2x$ . Then, if we move in the direction of increasing  $y$ , we enter the region through  $y = 0$  and leave through  $y = 1$ .

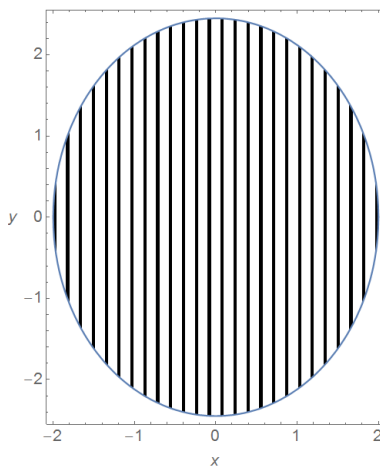
- So the bounds are  $0 \leq x \leq 1$ ,  $x \leq z \leq 2x$ , and  $0 \leq y \leq 1$  and thus the integral is  $\int_0^1 \int_x^{2x} \int_0^1 x^2 dy dz dx$ :

$$\begin{aligned} \int_0^1 \int_x^{2x} \int_0^1 x^2 dy dz dx &= \int_0^1 \int_x^{2x} [x^2 y] \Big|_{y=0}^1 dz dx = \int_0^1 \int_x^{2x} x^2 dz dx \\ &= \int_0^1 [x^2 z] \Big|_{z=x}^{2x} dx = \int_0^1 x^3 dx = \boxed{\frac{1}{4}}. \end{aligned}$$

- **Example:** Set up, but do not evaluate, an iterated triple integral for  $\iiint_D x^2 dV$ , where  $D$  is the region below the surface  $z = 12 - x^2 - y^2$  and above the surface  $z = 2x^2 + y^2$ .
  - Each of the surfaces is a parabolic bowl (the first opening downward and the second opening upward); here is a plot of each surface along with the region  $D$ :



- Since both surfaces are of the form  $z = f(x, y)$ , it will be easiest to project into the  $xy$ -plane. We can see (per the pictures) that the widest part of the solid occurs when the two surfaces intersect. We see that  $z = 12 - x^2 - y^2$  intersects  $z = 2x^2 + y^2$  when  $12 - x^2 - y^2 = 2x^2 + y^2$ , which is equivalent to  $3x^2 + 2y^2 = 12$ .
- Thus, we see that the projection of the solid into the  $xy$ -plane is the interior of the ellipse  $3x^2 + 2y^2 = 12$ , pictured below:



- With vertical slices (corresponding to the integration order  $dy dx$ ), we see that the corresponding range is  $-2 \leq x \leq 2$ ,  $-\sqrt{12 - 3x^2}/2 \leq y \leq \sqrt{12 - 3x^2}/2$ .
- Furthermore, since the upper surface is  $z = 12 - x^2 - y^2$  and the lower surface is  $z = 2x^2 + y^2$ , the range for  $z$  is  $2x^2 + y^2 \leq z \leq 12 - x^2 - y^2$ .

- Thus, the full integral is  $\int_{-2}^2 \int_{-\sqrt{12-3x^2}/2}^{\sqrt{12-3x^2}/2} \int_{2x^2+y^2}^{12-x^2-y^2} x^2 dz dy dx$ . (The value is  $10\pi\sqrt{3}$ , but the actual computation required to obtain this value involves a lot of algebra!)

- As a final remark, we will note that there exists a computational algorithm known as the cylindrical algebraic decomposition for converting a description of a region in space bounded by polynomial inequalities (such as  $x^2 \leq y + z$  or  $x^2 + y^2 + z^2 < 4$ ) to a union of regions described as  $a \leq x \leq b$ ,  $c(x) \leq y \leq d(x)$ ,  $e(x, y) \leq z \leq f(x, y)$ .
  - This algorithm, implemented in some computer algebra systems, provide a method for setting up integration problems over regions of these forms.
  - For example, applying this algorithm to the region defined by the inequalities  $x^2 + y^2 < 1$ ,  $x^2 + z^2 < 1$ ,  $y^2 + z^2 < 1$ , and  $0 < x < y < z$  shows that it consists of a single piece defined by  $0 < x < \frac{1}{\sqrt{2}}$ ,  $x < y < \frac{1}{\sqrt{2}}$ , and  $y < z < \sqrt{1 - y^2}$ .

### 3.3 Alternative Coordinate Systems and Changes of Variable

- We might wonder if there is a multivariable equivalent of the one-variable integration technique of substitution. If we think about substitution as being a “change of variables” to a new system of coordinates, then the answer is yes: we can rewrite multiple integrals in different coordinate systems.
- In order to simplify the computation of integrals on regions that have circular or spherical symmetries, which arise very often in physics due to the spherical symmetry of gravitational and electrical fields, there are several alternative coordinate systems we frequently employ beyond our usual “rectangular”  $xy$  and  $xyz$  coordinate systems: polar coordinates  $(r, \theta)$  in 2-dimensional space, and cylindrical  $(r, \theta, z)$  coordinates and spherical coordinates  $(\rho, \theta, \varphi)$  in 3-dimensional space.

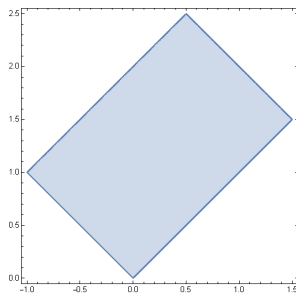
#### 3.3.1 General Changes of Variable

- Here are the general theorems on changing coordinates in a double or triple integral (we will not prove these):
- **Theorem** (General Substitution, 2 variables): If  $f(x, y)$  is continuous on  $R$ , and  $x = x(s, t)$  and  $y = y(s, t)$  are functions of  $s$  and  $t$ , then  $\iint_R f(x, y) dy dx = \iint_{R'} g(s, t) \left| \frac{\partial(x, y)}{\partial(s, t)} \right| dt ds$ , where  $R'$  is the region  $R$  expressed in  $st$ -coordinates,  $g(s, t) = f(x(s, t), y(s, t))$ , and  $\frac{\partial(x, y)}{\partial(s, t)} = J(x, y) = \begin{vmatrix} \partial x / \partial s & \partial x / \partial t \\ \partial y / \partial s & \partial y / \partial t \end{vmatrix}$  is the “Jacobian” of the coordinate transformation.
- **Theorem** (General Substitution, 3 variables): If  $f(x, y, z)$  is continuous on  $D$ , and  $x = x(s, t, u)$ ,  $y = y(s, t, u)$ , and  $z = z(s, t, u)$  are functions of  $s, t, u$ , then  $\iiint_D f(x, y, z) dz dy dx = \iiint_{D'} g(s, t, u) \left| \frac{\partial(x, y, z)}{\partial(s, t, u)} \right| du dt ds$ , where  $D'$  is the region  $D$  expressed in  $stu$ -coordinates,  $g(s, t, u) = f(x(s, t, u), y(s, t, u), z(s, t, u))$ , and  $\frac{\partial(x, y, z)}{\partial(s, t, u)} = J(x, y, z) = \begin{vmatrix} \partial x / \partial s & \partial x / \partial t & \partial x / \partial u \\ \partial y / \partial s & \partial y / \partial t & \partial y / \partial u \\ \partial z / \partial s & \partial z / \partial t & \partial z / \partial u \end{vmatrix}$  is the “Jacobian” of the coordinate transformation.
  - **Warning:** For this theorem to apply, the change of coordinates needs to be injective (one-to-one) on the original region of integration: if the image of the old region in the new set of coordinates ranges over some parts of the new region of integration more than once, the formula will be incorrect<sup>1</sup>.
  - The way in which the region of integration and the function transform under a change of coordinates are very natural. The only significant content of the substitution theorems are the statements about how the “differential of area”  $dA = dy dx$  and the “differential of volume”  $dV = dz dy dx$  transform.
  - The new differential is related to the old one by the Jacobian, which measures how the size of a small box in  $xy$  (or  $xyz$ )-coordinates changes when we switch to the new  $st$  (or  $stu$ )-coordinates. Giving a precise proof requires the geometric definition of the determinant of a matrix, which in fact measures precisely how a (linear) transformation changes areas/volumes.

<sup>1</sup>For example, the change of coordinates  $s = \sin(x)$ ,  $t = \sin(y)$  fails this criterion if applied to the entire  $xy$ -plane, because as  $x$  and  $y$  range over, say, the square  $0 \leq x \leq 4\pi$ ,  $0 \leq y \leq 4\pi$ , every point in the image square  $-1 \leq s \leq 1$ ,  $-1 \leq t \leq 1$  will be covered four times.

- Example: Evaluate the integral  $\iint_R (y+x) dA$  where  $R$  is the region bounded by the lines  $y = x$ ,  $y = 2 + x$ ,  $y = -x$ , and  $y = 3 - x$ .

- The region is a parallelogram:

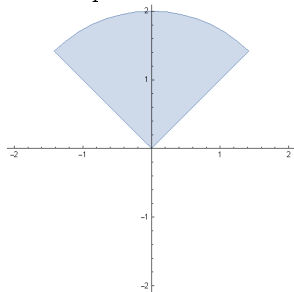


- In principle, we could divide this region into three pieces (using horizontal or vertical slices) and then compute the integral separately on each one. This would be rather lengthy since we would need to compute all of the points of intersection, then find the necessary bounds to describe each region, and so on.
- We will do it a different way: notice that the bounding lines can be written as  $y - x = 0$ ,  $y - x = 2$ ,  $y + x = 0$ , and  $y + x = 3$ .
- This suggests making the change of variables  $s = y + x$ ,  $t = y - x$ , since in  $st$ -coordinates the equations become the much simpler  $t = 0$ ,  $t = 2$ ,  $s = 0$ , and  $s = 3$ . (This change of variables is one-to-one since it is linear.)
- The new region is then  $0 \leq t \leq 2$ ,  $0 \leq s \leq 3$ , and the new function is  $x + y = s$ .
- For the Jacobian, we have  $x = \frac{s-t}{2}$  and  $y = \frac{s+t}{2}$ , so  $J(s, t) = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$ .
- The desired integral is then  $\iint_R (y+x) dA = \int_0^2 \int_0^3 s \cdot \frac{1}{2} ds dt = \int_0^2 \frac{9}{4} dt = \boxed{\frac{9}{2}}$ .

### 3.3.2 Double Integrals in Polar Coordinates

- Polar coordinates may already be familiar, as they are often discussed in precalculus and single-variable calculus.
- Definition: The polar coordinates  $(r, \theta)$  of a point  $(x, y)$  satisfy  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ , for  $r \geq 0$  and  $0 \leq \theta \leq 2\pi$ .
  - We have  $r = \sqrt{x^2 + y^2}$  and  $\theta = \tan^{-1}(y/x)$ , possibly plus  $\pi$  depending on the signs of  $x$  and  $y$ .
  - The parameter  $r$  is a distance giving the “radius” from the origin, and the parameter  $\theta$  measures the angle with respect to the positive  $x$ -axis.
  - Polar coordinates are useful in simplifying regions that are circular: the circle  $x^2 + y^2 = a^2$  in rectangular coordinates (over which it is cumbersome to set up double integrals) becomes the much simpler equation  $r = a$  in polar coordinates.
  - Polar coordinates are also useful in simplifying functions which involve  $x^2 + y^2$  or (especially)  $\sqrt{x^2 + y^2}$ .
  - Lines through the origin also have reasonably simple descriptions: the line  $y = mx$  becomes the pair of rays  $\theta = \tan^{-1}(m)$  and  $\theta = \tan^{-1}(m) + \pi$  when written in polar coordinates. (The two rays point in opposite directions.)
- For polar coordinates, we have  $J = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$ .
- Thus the differential in polar coordinates is  $dA = r dr d\theta$ . We typically set up polar integrals with this integration order, because most of the “nice” curves in polar coordinates have equation  $r = f(\theta)$  for some function  $f$ .
- Example: Integrate the function  $f(x, y) = x + 2y$  on the region  $R$  lying above the lines  $y = x$  and  $y = -x$  and inside the circle  $x^2 + y^2 = 4$ .

- Upon sketching the region, we see that it is a quarter-disc:



- In principle we could do this integral in rectangular coordinates, but it would be messy: using either horizontal or vertical slices would require splitting the region into two pieces, since the boundary curves change partway through the region in each case. So instead we will use polar coordinates.
- To describe the region, remember that in polar coordinates the line  $y = mx$  becomes the pair of rays  $\theta = \tan^{-1}(m)$  and  $\theta = \tan^{-1}(m) + \pi$ . (The two rays point in opposite directions.) The circle  $x^2 + y^2 = 4$  also becomes  $r = 2$ .
- Thus, the line  $y = x$  gives the right boundary  $\theta = \pi/4$  and the line  $y = -x$  gives the left boundary  $\theta = 3\pi/4$ . (The other two rays do not bound the region we are interested in.)
- So we see that the range for  $\theta$  is  $\pi/4 \leq \theta \leq 3\pi/4$ , and the range for  $r$  is  $0 \leq r \leq 2$ .
- Also, the function is  $f = x + 2y = r \cos \theta + 2r \sin \theta$ , and the differential is  $r dr d\theta$ .

- Thus, in polar coordinates the integral becomes  $\int_{\pi/4}^{3\pi/4} \int_0^2 (r \cos \theta + 2r \sin \theta) \cdot r dr d\theta$ , so we get

$$\begin{aligned} \int_{\pi/4}^{3\pi/4} \int_0^2 (r \cos \theta + 2r \sin \theta) \cdot r dr d\theta &= \int_{\pi/4}^{3\pi/4} (\cos \theta + 2 \sin \theta) \cdot \frac{1}{3} r^3 \Big|_{r=0}^2 d\theta \\ &= \int_0^{2\pi} \frac{8}{3} (\cos \theta + 2 \sin \theta) d\theta \\ &= \frac{8}{3} (-\sin \theta + 2 \cos \theta) \Big|_{\theta=\pi/4}^{3\pi/4} = \boxed{\frac{16\sqrt{2}}{3}}. \end{aligned}$$

- **Example:** Evaluate the integral  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}} dy dx$ .

- As written the integral is completely intractable, so we will try switching to polar coordinates.
- The region of integration is defined by the inequalities  $-1 \leq x \leq 1$ ,  $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ , which we can recognize as the interior of the unit circle  $x^2 + y^2 \leq 1$ .
- In polar coordinates, the region is  $0 \leq r \leq 1$  and  $0 \leq \theta \leq 2\pi$  (since there are no restrictions on  $\theta$ ).
- Also, the function is  $f(r, \theta) = e^{\sqrt{x^2+y^2}} = e^r$ , and the differential is  $r dr d\theta$ .
- Thus, in polar coordinates the integral becomes  $\int_0^{2\pi} \int_0^1 e^r \cdot r dr d\theta$ .
- Now we can evaluate it (using integration by parts to compute the inner integral in  $r$ ):

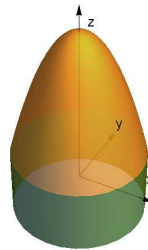
$$\int_0^{2\pi} \int_0^1 e^r \cdot r dr d\theta = \int_0^{2\pi} [r e^r - e^r] \Big|_{r=0}^1 d\theta = \int_0^{2\pi} 1 d\theta = \boxed{2\pi}.$$

- As an application of integration in polar coordinates, we can describe how to evaluate the famous Gaussian integral  $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ , which is quite difficult to compute because the function  $e^{-x^2}$  does not have an elementary antiderivative.
  - This integral is fundamental in statistics, since the function  $p(x) = e^{-x^2}$  arises (after a suitable change of variables) as the probability density function of the famous Gaussian normal distribution, which describes the distributions of quantities arising as the sum of independent small variations, such as human heights, errors in measurements, exam grades, and many other physical phenomena.

- To compute this integral, we note that if  $I = \int_{-\infty}^{\infty} e^{-x^2} dx$  then  $I = \int_{-\infty}^{\infty} e^{-y^2} dy$ , and so  $I^2 = \left[ \int_{-\infty}^{\infty} e^{-x^2} dx \right] \left[ \int_{-\infty}^{\infty} e^{-y^2} dy \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx$ .
- Now convert to polar coordinates: the region is the entire plane, with integration bounds  $0 \leq \theta \leq 2\pi$  and  $0 \leq r < \infty$ , and the function becomes  $e^{-(x^2+y^2)} = e^{-r^2}$ .
- Thus, in polar coordinates we see  $I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta$ , which we can now evaluate using a substitution  $u = r^2$  to see  $I^2 = \int_0^{2\pi} \left[ \frac{1}{2} e^{-r^2} \right]_{r=0}^{\infty} d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$ . Therefore, since  $I > 0$ , we deduce that  $I = \sqrt{\pi}$ .

### 3.3.3 Triple Integrals in Cylindrical Coordinates

- Cylindrical coordinates are a simple three-dimensional version of polar coordinates: we merely include the  $z$ -coordinate along with the polar coordinates  $r$  and  $\theta$ .
- Definition: The cylindrical coordinates  $(r, \theta, z)$  of a point  $(x, y, z)$  satisfy  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ ,  $z = z$  for  $r \geq 0$  and  $0 \leq \theta \leq 2\pi$ .
  - We have  $r = \sqrt{x^2 + y^2}$  and  $\theta = \tan^{-1}(y/x)$  (possibly plus  $\pi$  depending on the signs of  $x$  and  $y$ ), and obviously  $z = z$ .
  - The interpretations of  $r$  and  $\theta$  are essentially the same as in polar coordinates:  $r$  measures the distance of a point to the  $z$ -axis and  $\theta$  measures the angle (in a horizontal plane) from the positive  $x$ -direction.
  - Cylindrical coordinates are useful in simplifying regions that have a circular symmetry. In particular, the cylinder  $x^2 + y^2 = a^2$  in 3-dimensional rectangular coordinates (over which it is cumbersome to set up double integrals) becomes the simpler equation  $r = a$  in cylindrical.
- For cylindrical coordinates, we have  $J = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$ .
- Thus the differential in cylindrical coordinates is  $dV = r dz dr d\theta$ . We typically set up cylindrical integrals with this integration order (since typically the  $z$ -bounds are the most complicated), although other orders are possible.
- Example: Integrate the function  $f(x, y, z) = \sqrt{x^2 + y^2}$  on the solid enclosed by the cylinder  $x^2 + y^2 = 4$ , bounded above by the paraboloid  $z = 5 - x^2 - y^2$ , and bounded below by  $z = -1$ .
  - The surfaces involved suggest using cylindrical coordinates, since they both have reasonably simple descriptions in that coordinate system, as does the function  $f$  itself. Here is a plot of the region:



- In cylindrical coordinates, the cylinder has equation  $r = 2$ , the upper paraboloid has equation  $z = 5 - r^2$ , and the lower plane is  $z = -1$ .
  - \* The points inside the cylinder satisfy  $0 \leq r \leq 2$ , the points below the paraboloid satisfy  $z \leq 5 - r^2$ , and the points above the plane satisfy  $z \geq -1$ .
  - \* Since there are no restrictions on  $\theta$ , putting all of this together indicates that the integration bounds are  $0 \leq \theta \leq 2\pi$ ,  $0 \leq r \leq 2$ ,  $-1 \leq z \leq 5 - r^2$ .
- Since  $\sqrt{x^2 + y^2} = r$ , the function is simply  $f(r, \theta, z) = r$ , and the cylindrical differential is  $r dz dr d\theta$ .

- The integral is therefore equal to

$$\begin{aligned} \int_0^{2\pi} \int_0^2 \int_{-1}^{5-r^2} r \cdot r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^2 [r^2 z]_{z=-1}^{5-r^2} \, dr \, d\theta = \int_0^{2\pi} \int_0^2 [6r^2 - r^4] \, dr \, d\theta = \\ &= \int_0^{2\pi} \left[ 2r^3 - \frac{1}{5}r^5 \right]_{r=0}^2 d\theta = \int_0^{2\pi} \left( 16 - \frac{32}{5} \right) d\theta \\ &= \int_0^{2\pi} \frac{48}{5} \, d\theta = \boxed{\frac{96\pi}{5}}. \end{aligned}$$

- **Example:** Evaluate the integral  $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 z \sqrt{x^2+y^2} \, dz \, dy \, dx$ .

- This is an iterated integral of the function  $f(x, y, z) = z\sqrt{x^2+y^2}$  over the solid region  $D$  defined by the inequalities  $0 \leq x \leq 2$ ,  $0 \leq y \leq \sqrt{4-x^2}$ ,  $\sqrt{x^2+y^2} \leq z \leq 2$ .
- Notice that the projection of  $D$  into the  $xy$ -plane is the region  $0 \leq x \leq 2$ ,  $0 \leq y \leq \sqrt{4-x^2}$ , which is a quarter-disc. This, along with the presence of  $\sqrt{x^2+y^2}$  in the  $z$ -limit and in the function, strongly suggest converting to cylindrical coordinates.
- In cylindrical coordinates, we can see that the  $xy$ -region becomes  $0 \leq r \leq 2$ ,  $0 \leq \theta \leq \pi/2$ . Also, the range for  $z$  becomes  $r \leq z \leq 2$ .
- Since  $\sqrt{x^2+y^2} = r$ , the function is simply  $f = zr$ , and the cylindrical differential is  $r \, dz \, dr \, d\theta$ .
- The integral is therefore equal to

$$\begin{aligned} \int_0^{\pi/2} \int_0^2 \int_r^2 zr \cdot r \, dz \, dr \, d\theta &= \int_0^{\pi/2} \int_0^2 \left[ \frac{1}{2}z^2r^2 \right]_{z=r}^2 \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 \left[ 2r^2 - \frac{1}{2}r^4 \right] \, dr \, d\theta = \\ &= \int_0^{\pi/2} \left[ \frac{2}{3}r^3 - \frac{1}{10}r^5 \right]_{r=0}^2 d\theta = \int_0^{\pi/2} \left( \frac{16}{3} - \frac{32}{10} \right) d\theta \\ &= \int_0^{\pi/2} \frac{32}{15} \, d\theta = \boxed{\frac{16\pi}{15}}. \end{aligned}$$

- **Example:** Integrate the function  $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2}}$  on the region underneath  $z = 9 - x^2 - y^2$  and above the  $xy$ -plane.

- We set up this problem in cylindrical coordinates: the paraboloid has equation  $z = 9 - r^2$ , so the portion with  $z \geq 0$  corresponds to  $0 \leq r \leq 3$ . There are no restrictions on  $\theta$ , so we have  $0 \leq \theta \leq 2\pi$ , and also  $0 \leq z \leq 9 - r^2$ .
- Since  $\sqrt{x^2+y^2} = r$ , the function is simply  $f = 1/r$ , and the cylindrical differential is  $r \, dz \, dr \, d\theta$ .
- The desired integral is thus  $\int_0^{2\pi} \int_0^3 \int_0^{9-r^2} 1 \, dz \, dr \, d\theta$ . Now we can evaluate it:

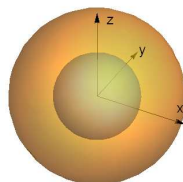
$$\int_0^{2\pi} \int_0^3 \int_0^{9-r^2} 1 \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^3 (9 - r^2) \, dr \, d\theta = \int_0^{2\pi} 18 \, d\theta = \boxed{36\pi}.$$

### 3.3.4 Triple Integrals in Spherical Coordinates

- Spherical coordinates are another 3-dimensional version of polar coordinates:
- **Definition:** The spherical coordinates  $(\rho, \theta, \varphi)$  of a point  $(x, y, z)$  satisfy  $x = \rho \sin \varphi \cos \theta$ ,  $y = \rho \sin \varphi \sin \theta$ ,  $z = \rho \cos \varphi$  for  $\rho \geq 0$ ,  $0 \leq \theta \leq 2\pi$ , and  $0 \leq \varphi \leq \pi$ .
  - With  $r$  and  $\theta$  as defined for cylindrical coordinates, we also have  $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$ ,  $r = \rho \sin \varphi$ , and  $\varphi = \tan^{-1}(r/z)$ .



- The parameter  $\rho$  measures the distance from the origin. The angle  $\theta$  has the same interpretation as in cylindrical coordinates, while the angle  $\varphi$  measures the angle with the positive  $z$ -axis (often called “declination from the  $z$ -axis”): thus, for example, the graph of  $\varphi = \pi/2$  consists of the points in the  $xy$ -plane, since the vectors joining these points with the origin make a right angle with the  $x$ -axis.
- **Important Notational Remark:** Most applied scientific fields that make use of spherical coordinates (e.g., chemistry and physics) have their own standard notations for spherical coordinates which differ from the standard used in most of mathematics.
  - \* Specifically, the angles  $\varphi$  and  $\theta$  are generally reversed from the above, and  $r$  and  $\rho$  are often reversed as well. In addition, the angle  $\varphi$  is sometimes measured as an inclination from the  $xy$ -plane rather than a declination from the  $z$ -axis, thus giving it a range of  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  rather than  $[0, \pi]$ .
  - \* In mathematics, we set up the angles this way in order to maintain consistency with polar coordinates, so that  $\theta$  means the same thing in both contexts.
- Spherical coordinates are most useful when integrating over regions with spherical symmetries: the sphere  $x^2 + y^2 + z^2 = a^2$ , over which it is typically very difficult to set up triple integrals, becomes the much simpler  $\rho = a$  in spherical coordinates.
- The cone  $az = \sqrt{x^2 + y^2}$  with vertex at the origin also has a simple expression in spherical coordinates, namely as  $\varphi = \tan^{-1}(a)$ .
- For spherical coordinates,  $J = \left| \frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} \right| = \begin{vmatrix} \cos \theta \sin \varphi & \rho \cos \theta \cos \varphi & -\rho \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & \rho \sin \theta \cos \varphi & \rho \cos \theta \sin \varphi \\ \cos \varphi & -\rho \sin \varphi & 0 \end{vmatrix} = \rho^2 \sin \varphi$ .
- Thus the differential in spherical coordinates is  $dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$ . We typically set up spherical integrals with this integration order, because typically the  $\rho$  bounds are the most complicated, while the  $\theta$  bounds are the simplest.
- **Example:** Integrate the function  $f(x, y, z) = z^2$  over the region  $1 \leq x^2 + y^2 + z^2 \leq 4$ .
  - Here is a plot of the region of integration, which is bounded by two spheres:



- Given that the region is bounded by the two spheres  $x^2 + y^2 + z^2 = 1$  and  $x^2 + y^2 + z^2 = 4$ , we will switch to spherical. The first sphere is  $\rho = 1$  and the second is  $\rho = 2$ , and there are no restrictions on the angles  $\varphi$  and  $\theta$ . Thus, the region of integration is  $1 \leq \rho \leq 2$ ,  $0 \leq \varphi \leq \pi$ , and  $0 \leq \theta \leq 2\pi$ .
- In spherical coordinates, the function is  $f(\rho, \theta, \varphi) = z^2 = \rho^2 \cos^2(\varphi)$  and the differential is  $\rho^2 \sin(\varphi) d\rho d\varphi d\theta$ .
- The integral in spherical coordinates is therefore  $\int_0^{2\pi} \int_0^\pi \int_1^2 \rho^4 \cos^2(\varphi) \sin(\varphi) d\rho d\varphi d\theta$ . Now we can evaluate it:

$$\begin{aligned}
 \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^4 \cos^2(\varphi) \sin(\varphi) d\rho d\varphi d\theta &= \int_0^{2\pi} \int_0^\pi \left[ \frac{\rho^5}{5} \cos^2(\varphi) \sin(\varphi) \right]_{\rho=1}^2 d\varphi d\theta \\
 &= \int_0^{2\pi} \int_0^\pi \frac{31}{5} \cos^2(\varphi) \sin(\varphi) d\varphi d\theta \\
 &= \int_0^{2\pi} \frac{31}{5} \cdot \left[ -\frac{1}{3} \cos^3(\varphi) \right]_{\varphi=0}^\pi d\theta \\
 &= \int_0^{2\pi} \frac{31}{5} \cdot \frac{2}{3} d\theta = \boxed{\frac{124}{15} \pi}.
 \end{aligned}$$

- Example: Evaluate the integral  $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2} dz dy dx$ .
  - This is an iterated integral of the function  $f(x, y, z) = z\sqrt{x^2+y^2}$  over the solid region  $D$  defined by the inequalities  $0 \leq x \leq 1$ ,  $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ ,  $\sqrt{x^2+y^2} \leq z \leq \sqrt{2-x^2-y^2}$ .
  - Notice that this region  $D$  is the region bounded by the planes  $x = 0$  and  $y = 0$  that lies below the hemisphere  $z = \sqrt{2-x^2-y^2}$  and above the cone  $z = \sqrt{x^2+y^2}$ . All of these surfaces have very simple descriptions in spherical coordinates, as does the function  $f(x, y, z) = \sqrt{x^2+y^2+z^2}$ .
  - Specifically, the hemisphere becomes  $\rho = \sqrt{2}$  and the cone becomes  $\varphi = \pi/4$ . Since we want the region above the cone, we need  $0 \leq \varphi \leq \pi/4$ , and since we want the region inside the hemisphere we want  $0 \leq \rho \leq \sqrt{2}$ . Also, since we want the region where  $x \geq 0$  and  $y \geq 0$ , we want  $0 \leq \theta \leq \pi/2$ .
  - Since  $\sqrt{x^2+y^2+z^2} = \rho$ , the function is simply  $f = \rho$ , and the differential is  $\rho^2 \sin(\varphi) d\rho d\varphi d\theta$ .
  - The integral in spherical coordinates is therefore  $\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho \cdot \rho^2 \sin(\varphi) d\rho d\varphi d\theta$ . Now we can evaluate it:

$$\begin{aligned}
 \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho \cdot \rho^2 \sin(\varphi) d\rho d\varphi d\theta &= \int_0^{\pi/2} \int_0^{\pi/4} \left[ \frac{\rho^4}{4} \sin(\varphi) \right] \Big|_{\rho=0}^{\sqrt{2}} d\varphi d\theta \\
 &= \int_0^{\pi/2} \int_0^{\pi/4} \sin(\varphi) d\varphi d\theta \\
 &= \int_0^{\pi/2} [-\cos(\varphi)] \Big|_{\varphi=0}^{\pi/4} d\theta \\
 &= \int_0^{\pi/2} \left( 1 - \frac{\sqrt{2}}{2} \right) d\theta = \boxed{\left( 1 - \frac{\sqrt{2}}{2} \right) \frac{\pi}{2}}.
 \end{aligned}$$

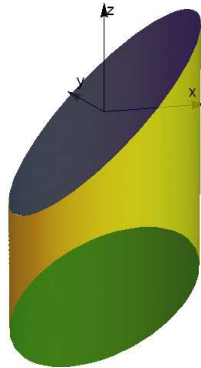
- Example: Integrate the function  $f(x, y, z) = \frac{z}{\sqrt{x^2+y^2}}$  over the lower half of the sphere  $x^2+y^2+z^2=9$ .
  - We set up this problem in spherical coordinates: the sphere is  $\rho = 3$ , and the lower half corresponds to  $\pi/2 \leq \varphi \leq \pi$ . Since there are no restrictions on  $\theta$ , the region of integration is  $0 \leq \rho \leq 3$ ,  $\pi/2 \leq \varphi \leq \pi$ , and  $0 \leq \theta \leq 2\pi$ .
  - In spherical coordinates, the function is  $f(\rho, \theta, \varphi) = \frac{\rho \cos(\varphi)}{\sqrt{\rho^2 \sin^2(\varphi)}} = \frac{\cos(\varphi)}{\sin(\varphi)}$  and the differential is  $\rho^2 \sin(\varphi) d\rho d\varphi d\theta$ .
  - The desired integral is thus  $\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^3 \rho^2 \cos(\varphi) d\rho d\varphi d\theta$ . Now we can evaluate it:

$$\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_0^3 \rho^2 \cos(\varphi) d\rho d\varphi d\theta = \int_0^{2\pi} \int_{\pi/2}^{\pi} 9 \cos(\varphi) d\varphi d\theta = \int_0^{2\pi} 9d\theta = \boxed{18\pi}.$$

### 3.3.5 Additional Examples of Coordinate Changes

- To change a multiple integral into a different coordinate system (polar, cylindrical, spherical, or something more general), follow these steps:
  1. Sketch the region of integration.
  2. Decide on the new coordinate system to use, and express the region in the new coordinate system.
    - Usually the choice will be decided by shape of the region of integration, or sometimes the function being integrated.

- For example, an integral over  $x^2 + y^2 + z^2 \leq 1$  is a good choice for spherical coordinates, since the region takes the much simpler form  $0 \leq \rho \leq 1$ , while an integral over the region  $x^2 + y^2 \leq 4$ ,  $-2 \leq z \leq 2$  is probably better in cylindrical coordinates.
  - In general, the best coordinate system is one that turns the region into something simple to describe, and that simplifies the function.
  - Cylinders  $x^2 + y^2 = a^2$  and other surfaces of the form  $z = f(x, y)$  are easiest to describe in cylindrical coordinates, while spheres  $x^2 + y^2 + z^2 = a^2$  are easiest in spherical coordinates. Cones of the form  $z = \alpha\sqrt{x^2 + y^2}$  have easy descriptions in both systems: as  $z = \alpha r$  and as  $\varphi = \tan^{-1}(\alpha)$  respectively.
3. Express the function in the new coordinates.
    - For polar and cylindrical, substitute  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ . (And  $z = z$ , in cylindrical.)
    - For spherical, substitute  $x = \rho \cos(\theta) \sin(\varphi)$ ,  $y = \rho \sin(\theta) \sin(\varphi)$ ,  $z = \rho \cos(\varphi)$ .
  4. Change the differential into the new coordinates. For a general transformation, this requires multiplying by the Jacobian, a determinant whose terms are  $\frac{\partial[\text{old}]}{\partial[\text{new}]}$ .
    - For polar the new differential is  $r dr d\theta$ , for cylindrical it is  $r dr d\theta dz$ , and for spherical, it is  $\rho^2 \sin(\varphi) d\rho d\varphi d\theta$ .
  5. Evaluate the new integral.
- **Example:** Integrate  $f(x, y, z) = \sqrt{x^2 + y^2}$  over the region inside  $x^2 + y^2 = 1$ , below the plane  $z = x$ , and above the plane  $z = -2 - y$ .
    - Here is a plot of the region:

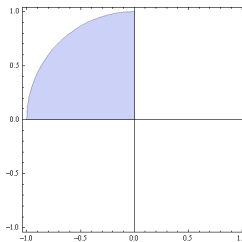


- The surface  $x^2 + y^2 = 1$  is a cylinder, and the other two bounding curves are functions of  $z$ . The function also involves  $x^2 + y^2$ ; all of these things indicate that we should switch to cylindrical coordinates.
- The inside of the cylinder is given by  $0 \leq r \leq 1$  in cylindrical coordinates.
- As we move in the direction of increasing  $z$ , we enter the region through the plane  $z = -2 - y = -2 - r \sin(\theta)$  and exit via the plane  $z = x = r \cos(\theta)$ , so the  $z$ -bounds are  $-2 - r \sin(\theta) \leq z \leq r \cos(\theta)$ .
- There are no restrictions on  $\theta$  so we have the simple bounds  $0 \leq \theta \leq 2\pi$ .
- In cylindrical coordinates, the function is  $f(x, y) = \sqrt{x^2 + y^2} = r$ , and the differential is  $r dz dr d\theta$ .
- Thus, in cylindrical coordinates the integral is  $\int_0^{2\pi} \int_0^1 \int_{-2-r \sin(\theta)}^{r \cos(\theta)} r \cdot r dz dr d\theta$ . Now we can evaluate it:

$$\begin{aligned}
 \int_0^{2\pi} \int_0^1 \int_{-2-r \sin(\theta)}^{r \cos(\theta)} r \cdot r dz dr d\theta &= \int_0^{2\pi} \int_0^1 r^2 [r \cos(\theta) - (-2 - r \sin(\theta))] dr d\theta \\
 &= \int_0^{2\pi} \int_0^1 [r^3 \cos(\theta) + 2r^2 + r^3 \sin(\theta)] dr d\theta \\
 &= \int_0^{2\pi} \left[ \frac{1}{4} \cos(\theta) + \frac{2}{3} + \frac{1}{4} \sin(\theta) \right] d\theta = \boxed{\frac{4\pi}{3}}.
 \end{aligned}$$

- Example: Evaluate the integral  $\int_0^1 \int_{-\sqrt{1-y^2}}^0 \frac{2}{(1+x^2+y^2)^2} dx dy$ .

- The region is defined by  $0 \leq y \leq 1$  and  $-\sqrt{1-y^2} \leq x \leq 0$ . Upon sketching, we see that it is a quarter-circle of radius 1:



- In theory, it is possible to evaluate the integral as written, but it would be rather unpleasant, so we look for an alternative.
- The  $-\sqrt{1-y^2}$  term in the limits of integration and the  $x^2 + y^2$  terms in the function strongly suggest using polar coordinates, as the limits in polar are very simple:  $\frac{\pi}{2} \leq \theta \leq \pi$  and  $0 \leq r \leq 1$ .

- To convert the function into polar coordinates, we substitute  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ : then  $\frac{2}{(1+x^2+y^2)^2} = \frac{2}{(1+r^2)^2}$ .

- Since the polar differential is  $r dr d\theta$ , the new integral is  $\int_{\pi/2}^{\pi} \int_0^1 \frac{2}{(1+r^2)^2} r dr d\theta$ .

- Now it is much easier to evaluate: we can substitute  $u = 1 + r^2$  with  $du = 2r dr$  to obtain

$$\int_{\pi/2}^{\pi} \int_0^1 \frac{2}{(1+r^2)^2} r dr d\theta = \int_{\pi/2}^{\pi} \int_1^2 \frac{1}{u^2} du d\theta = \int_{\pi/2}^{\pi} [u^{-1}]_{u=1}^2 d\theta = \int_{\pi/2}^{\pi} \frac{1}{2} d\theta = \boxed{\frac{\pi}{4}}.$$

- Example: Integrate the function  $f(x, y, z) = 1$  on the region inside  $x^2 + y^2 + z^2 = 9$ , below  $z = \sqrt{x^2 + y^2}$ , and above the  $xy$ -plane.

- The first surface  $x^2 + y^2 + z^2 = 9$  is a sphere of radius 3, and the second surface is a cone. Spheres and cones have simple descriptions in spherical coordinates, so we will use spherical coordinates.

- In spherical coordinates, the equation  $x^2 + y^2 + z^2 = 9$  becomes  $\rho = 3$ , the equation  $z = \sqrt{x^2 + y^2}$  becomes  $\rho \cos \varphi = \rho \sin \varphi$  which is easily seen to be the same as  $\varphi = \frac{\pi}{4}$ , and the  $xy$ -plane has equation  $\varphi = \frac{\pi}{2}$ .

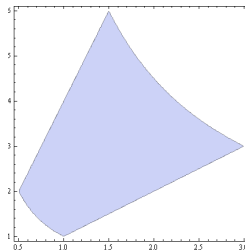
- The bounds of integration in spherical coordinates are therefore  $0 \leq \theta \leq 2\pi$ ,  $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$ ,  $0 \leq \rho \leq 3$ .

- Furthermore, the function is simply 1, and the differential is  $\rho^2 \sin \varphi d\rho d\varphi d\theta$ .

- The integral is therefore  $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^3 \rho^2 \sin \varphi d\rho d\varphi d\theta = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} 9 \sin \varphi d\varphi d\theta = \int_0^{2\pi} 9 \frac{\sqrt{2}}{2} d\theta = \boxed{9\pi\sqrt{2}}$ .

- Example: Evaluate  $\iint_R f(x, y) dA$ , where  $f(x, y) = x^2 + y^2$ , and  $R$  is the region in the first quadrant defined by the inequalities  $1 \leq xy \leq 9$ ,  $1 \leq \frac{y}{x} \leq 4$ .

- The region is as follows:



- We could in principle divide the region into three pieces (horizontally or vertically), but this would be somewhat cumbersome.
- Instead, based on the bounds of integration, we will make a change of variables to set  $s = xy$ ,  $t = \frac{y}{x}$ .

Solving for  $x$  and  $y$  in terms of  $s$  and  $t$  yields  $x = \sqrt{\frac{s}{t}}$  and  $y = \sqrt{st}$ .

- The new bounds of integration are  $1 \leq s \leq 9$  and  $1 \leq t \leq 4$ , and the new function is  $x^2 + y^2 = \frac{s}{t} + st$ .
- For the new differential, we compute the Jacobian:  $J = \begin{vmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} \frac{1}{2}s^{-1/2}t^{-1/2} & -\frac{1}{2}s^{1/2}t^{-3/2} \\ \frac{1}{2}s^{-1/2}t^{1/2} & \frac{1}{2}s^{1/2}t^{-1/2} \end{vmatrix} = \left(\frac{1}{2}s^{-1/2}t^{-1/2}\right)\left(\frac{1}{2}s^{1/2}t^{-1/2}\right) - \left(\frac{1}{2}s^{-1/2}t^{1/2}\right)\left(-\frac{1}{2}s^{1/2}t^{-3/2}\right) = \frac{1}{2t}$ . Hence the new differential is  $\frac{1}{2t} dt ds$ .
- The integral in  $st$ -coordinates is thus  $\int_0^9 \int_1^4 \left(\frac{s}{t} + st\right) \cdot \left(\frac{1}{2t}\right) dt ds$ . Now we evaluate:

$$\begin{aligned} \int_1^9 \int_1^4 \left(\frac{s}{t} + st\right) \cdot \left(\frac{1}{2t}\right) dt ds &= \int_1^9 \int_1^4 \frac{s}{2} [t^{-2} + 1] dt ds = \int_1^9 \frac{s}{2} [-t^{-1} + t] \Big|_{t=1}^4 ds \\ &= \int_1^9 \frac{s}{2} \left[\frac{3}{4} + 3\right] ds = \int_1^9 \frac{15}{8} s ds = \boxed{75}. \end{aligned}$$

- **Example:** Evaluate the integral  $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)^{-3/4} dz dy dx$ .

- The region is defined by the inequalities  $0 \leq x \leq 2$ ,  $0 \leq y \leq \sqrt{4-x^2}$ , and  $0 \leq z \leq \sqrt{4-x^2-y^2}$ , which collectively describe the region inside the sphere  $x^2 + y^2 + z^2 = 4$  inside the first octant (i.e., with  $x, y, z$  nonnegative).
- Furthermore, the integrand has a very simple description in spherical coordinates, since it is simply  $(\rho^2)^{-3/4} = \rho^{-3/2}$ .
- Therefore, we can evaluate this integral by converting to spherical coordinates. The region is defined by  $0 \leq \theta \leq \pi/2$ ,  $0 \leq \varphi \leq \pi/2$ , and  $0 \leq \rho \leq 2$ , while the function is  $\rho^{-5}$ .
- Thus, in spherical coordinates, the integral is

$$\begin{aligned} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^{-3/2} \cdot \rho^2 \sin \varphi d\rho d\varphi d\theta &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^{1/2} \sin \varphi d\rho d\varphi d\theta \\ &= \int_0^{\pi/2} \int_0^{\pi/2} \frac{4\sqrt{2}}{3} \sin \varphi d\varphi d\theta = \int_0^{\pi/2} \frac{4\sqrt{2}}{3} = \boxed{2\pi\sqrt{2}/3}. \end{aligned}$$

- **Example:** Evaluate the integral  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{\sqrt{x^2+y^2}} \frac{z}{\sqrt{x^2+y^2}} dz dy dx$ .

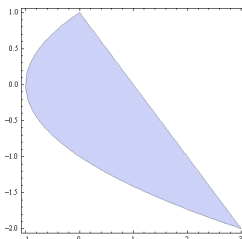
- The region is defined by the inequalities  $-1 \leq x \leq 1$ ,  $-\sqrt{1-x^2} \leq y \leq 0$ , and  $0 \leq z \leq \sqrt{x^2+y^2}$ . The  $x$  and  $y$  inequalities describe the lower half of the interior of the unit circle, while the  $z$  inequalities have simple descriptions in cylindrical coordinates, as does the integrand.
- Therefore, we can evaluate this integral by converting to cylindrical coordinates. The region is defined by  $\pi \leq \theta \leq 2\pi$ ,  $0 \leq r \leq 1$ , and  $0 \leq z \leq r$ , while the function is  $\frac{z}{r}$ .
- Thus, the integral is  $\int_{\pi}^{2\pi} \int_0^1 \int_0^r \frac{z}{r} dz dr d\theta = \int_{\pi}^{2\pi} \int_0^1 \frac{1}{2} r^2 dr d\theta = \int_{\pi}^{2\pi} \frac{1}{6} d\theta = \boxed{\pi/6}$ .

### 3.4 Applications of Multiple Integration

- There are a number of applications of multiple integration to computing quantities like areas and volumes, the average value of a function on a region, and the mass of a lamina (a solid plate with variable density) in the plane or of a solid having variable density in space.

### 3.4.1 Areas, Volumes, and Average Values

- One straightforward but still very useful application of multiple integration is to computing areas of regions in the plane, and volumes of regions in space.
  - The central ideas are that  $\text{Area}(R) = \iint_R 1 \, dA$  and  $\text{Volume}(D) = \iiint_D 1 \, dV$ .
  - Thus, if we can describe a region in a form that lends itself to integration, we can calculate the region's area (or volume).
  - A closely related problem is to calculate the average value of a function on a region. To calculate this, we simply integrate the function over the region, and then divide by the region's area or volume.
- Example: Find the area of the region bounded by the curves  $x = y^2 - 1$  and  $y = 1 - x$ . Then find the average value of  $y$  on this region.
  - A sketch of the region indicates its shape is as below:



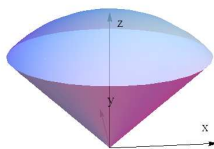
- We can set up the area integral with either integration order, but if we use vertical slices we would need to divide the region into two pieces, since the top curve changes from the upper half of the parabola to the line in the middle of the region of integration.
- Therefore, we use horizontal slices with the integration order  $dx \, dy$ : we see that the range for  $y$  is  $-2 \leq y \leq 1$ , and then the corresponding range for  $x$  is  $y^2 - 1 \leq x \leq 1 - y$ .
- The area is then given by  $\int_{-2}^1 \int_{y^2-1}^{1-y} 1 \, dx \, dy$ , which we can evaluate:

$$\begin{aligned} \int_{-2}^1 \int_{y^2-1}^{1-y} 1 \, dx \, dy &= \int_{-2}^1 [x] \Big|_{x=y^2-1}^{1-y} \, dy = \int_{-2}^1 [(1-y) - (y^2-1)] \, dy \\ &= \int_{-2}^1 [2-y-y^2] \, dy = \left[ 2y - \frac{1}{2}y^2 - \frac{1}{3}y^3 \right] \Big|_{y=-2}^1 = \boxed{\frac{9}{2}}. \end{aligned}$$

- Now to calculate the average value of  $y$ , we evaluate  $\frac{1}{\text{Area}(R)} \cdot \int_{-2}^1 \int_{y^2-1}^{1-y} y \, dx \, dy$ :

$$\begin{aligned} \frac{1}{\text{Area}(R)} \cdot \int_{-2}^1 \int_{y^2-1}^{1-y} y \, dx \, dy &= \frac{2}{9} \int_{-2}^1 [xy] \Big|_{x=y^2-1}^{1-y} \, dy = \frac{2}{9} \int_{-2}^1 [y(1-y) - y(y^2-1)] \, dy \\ &= \frac{2}{9} \int_{-2}^1 [2y - y^2 - y^3] \, dy = \frac{2}{9} \left[ y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4 \right] \Big|_{y=-2}^1 = \boxed{-\frac{1}{2}}. \end{aligned}$$

- Example: Find the volume of the “ice cream cone” solid that lies inside the sphere  $x^2 + y^2 + z^2 = 1$  and above the cone  $z = \sqrt{x^2 + y^2}$ . Then find the average value of  $z$  on this region.
  - Here is the plot of the solid:



- Since we are dealing with a sphere, we set up the problem in spherical coordinates.

- The sphere has equation  $\rho = 1$ , and the interior is  $0 \leq \rho \leq 1$ .
- For the cone, we know  $\sqrt{x^2 + y^2} = \rho \sin \varphi$  and  $z = \rho \cos \varphi$ , so the cone  $z = \sqrt{x^2 + y^2}$  in spherical is the same as  $\sin \varphi = \cos \varphi$ , or  $\varphi = \frac{\pi}{4}$ . Since  $\varphi$  measures declination from the  $z$ -axis, the region above the cone is therefore  $0 \leq \varphi \leq \frac{\pi}{4}$ .

- Hence the volume of this region is given by  $\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 1 \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$ . We evaluate:

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 1 \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3} \sin \varphi \, d\varphi \, d\theta = \int_0^{2\pi} \frac{1}{3} \left(1 - \frac{\sqrt{2}}{2}\right) \, d\theta = \boxed{\frac{\pi(2 - \sqrt{2})}{3}}.$$

- Now to calculate the average value of  $z = \rho \cos \phi$ , we evaluate  $\frac{1}{\text{Volume}(D)} \cdot \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho \cos \varphi \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$ :

$$\begin{aligned} \frac{1}{\text{Volume}(D)} \cdot \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho \cos \varphi \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta &= \frac{3}{\pi(2 - \sqrt{2})} \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \frac{1}{2} \rho^3 \sin(2\varphi) \, d\rho \, d\varphi \, d\theta \\ &= \frac{3}{\pi(2 - \sqrt{2})} \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{8} \sin(2\varphi) \, d\varphi \, d\theta \\ &= \frac{3}{\pi(2 - \sqrt{2})} \int_0^{2\pi} \frac{1}{16} \, d\theta = \boxed{\frac{3}{8(2 - \sqrt{2})}}. \end{aligned}$$

### 3.4.2 Masses, Centroids, and Moments of a Plate and Solid

- The center of mass of a physical object is its “balancing point”, where, if the object is supported only at that point, gravity will not cause it to tip over. The center of mass is also called the centroid of an object.
- If we are given the shape and density of an object in 2 or 3 dimensions, then we can find its mass and the location of its center of mass. We can also calculate its “moments” about an axis or a point, which show up in physics calculations involving angular momentum. Each of these quantities is the integral of a simple function, “weighted” by the density of the object.
- Center of Mass and Moment Formulas (2 Dimensions): We are given a 2-dimensional plate of variable density  $\delta(x, y)$  on a region  $R$ .
  - The total mass  $M$  is given by  $M = \iint_R \delta(x, y) \, dA$ .
  - The  $x$ -moment  $M_y$  is given by  $M_y = \iint_R x \delta(x, y) \, dA$ .
  - The  $y$ -moment  $M_x$  is given by  $M_x = \iint_R y \delta(x, y) \, dA$ .
    - \* The notation for these moments is rather confusing in this instance: the phrase “ $x$ -moment” means the moment of the object about the  $x$ -axis, which is given by integrating the *distance* to the  $x$ -axis (namely,  $y$ ) rather than the  $x$ -coordinate.
  - The center of mass  $(\bar{x}, \bar{y})$  has coordinates  $\left(\frac{M_y}{M}, \frac{M_x}{M}\right)$ .
    - \* Equivalently, each coordinate of the centroid is given by the average value of that coordinate over the plate – hence the notation  $\bar{x}$ , which means “the average value of  $x$ ”.
- Center of Mass and Moment Formulas (3 Dimensions): We are given a 3-dimensional plate of variable density  $\delta(x, y, z)$  on a region  $D$ .
  - The total mass  $M$  is given by  $M = \iiint_D \delta(x, y, z) \, dV$ .
  - The moment  $M_{yz}$  about the  $yz$ -plane is given by  $M_{yz} = \iiint_D x \delta(x, y, z) \, dV$ .
  - The moment  $M_{xz}$  about the  $xz$ -plane is given by  $M_{xz} = \iiint_D y \delta(x, y, z) \, dV$ .
  - The moment  $M_{xy}$  about the  $xy$ -plane is given by  $M_{xy} = \iiint_D z \delta(x, y, z) \, dV$ .

- The center of mass  $(\bar{x}, \bar{y}, \bar{z})$  has coordinates  $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$ .
- Example: Find the mass and center of mass of a plate on the triangle  $0 \leq x \leq 1, 0 \leq y \leq x$  whose density is  $\delta(x, y) = y^2$ .

- The total mass  $M$  is given by  $M = \iint_R \delta(x, y) dA = \int_0^1 \int_0^x y^2 dy dx = \int_0^1 \frac{1}{3} x^3 dx = \boxed{\frac{1}{12}}$ .

- The  $x$ -coordinate of the centroid is  $\frac{1}{M} \iint_R x \delta(x, y) dA = 12 \int_0^1 \int_0^x xy^2 dy dx = 12 \int_0^1 \frac{1}{3} x^4 dx = \frac{12}{15}$ .

- The  $y$ -coordinate of the centroid is  $\frac{1}{M} \iint_R y \delta(x, y) dA = 12 \int_0^1 \int_0^x y^3 dy dx = 12 \int_0^1 \frac{1}{4} x^4 dx = \frac{12}{20}$ .

- Therefore, the center of mass has coordinates  $(\bar{x}, \bar{y}) = \boxed{\left(\frac{4}{5}, \frac{3}{5}\right)}$ .

- Example: Find the mass and center of mass of a solid in the shape of the cube bounded  $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ , whose density is  $\delta(x, y, z) = 1 + xyz$ .

- The total mass  $M$  is given by

$$M = \iiint_D \delta(x, y, z) dV = \int_0^1 \int_0^1 \int_0^1 (1 + xyz) dz dy dx = \int_0^1 \int_0^1 \left(1 + \frac{1}{2}xy\right) dy dx = \int_0^1 \left(1 + \frac{1}{4}x\right) dx = \boxed{\frac{9}{8}}.$$

- The  $x$ -coordinate of the centroid is given by

$$\frac{1}{M} \iiint_D x \delta(x, y, z) dV = \frac{8}{9} \int_0^1 \int_0^1 \int_0^1 (x + x^2yz) dz dy dx = \frac{8}{9} \int_0^1 \int_0^1 \left(x + \frac{1}{2}x^2y\right) dy dx = \frac{8}{9} \int_0^1 \left(x + \frac{1}{4}x^2\right) dx = \boxed{\frac{21}{32}}.$$

- The  $y$ -coordinate of the centroid is given by

$$\frac{1}{M} \iiint_D y \delta(x, y, z) dV = \frac{8}{9} \int_0^1 \int_0^1 \int_0^1 (y + xy^2z) dz dy dx = \frac{8}{9} \int_0^1 \int_0^1 \left(y + \frac{1}{2}xy^2\right) dy dx = \frac{8}{9} \int_0^1 \left(\frac{1}{2} + \frac{1}{6}x\right) dx = \boxed{\frac{21}{32}}.$$

- The  $z$ -coordinate of the centroid is given by

$$\frac{1}{M} \iiint_D z \delta(x, y, z) dV = \frac{8}{9} \int_0^1 \int_0^1 \int_0^1 (z + xyz^2) dz dy dx = \frac{8}{9} \int_0^1 \int_0^1 \left(\frac{1}{2} + \frac{1}{3}xy\right) dy dx = \frac{8}{9} \int_0^1 \left(\frac{1}{2} + \frac{1}{6}x\right) dx = \boxed{\frac{21}{32}}.$$

- Therefore, the center of mass has coordinates  $(\bar{x}, \bar{y}, \bar{z}) = \boxed{\left(\frac{21}{32}, \frac{21}{32}, \frac{21}{32}\right)}$ .

Well, you're at the end of my handout. Hope it was helpful.

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