Contents

1   Vectors and 3-Dimensional Geometry

From one-variable calculus, we are very used to 2-dimensional "planar" geometry: given a function $f(x)$, we know how to produce the graph of the curve $y = f(x)$ and use calculus to identify some of its important features such as local minima and maxima. We also know how to graph parametrically-defined functions such as the cycloid $x = t + \sin(t)$, $y = 1 + \cos(t)$. And we even know, more or less, how to graph implicitly-defined functions, like the circle $x^2 + y^2 = 1$.

In this chapter, we introduce 3-dimensional space (often called 3-space for short) and study the geometry of lines and planes, as well as more general curves and surfaces, using functions of several variables. Along the way, we will introduce and study vectors, which will help to clarify a great deal of the concepts.

1.1   Functions of Several Variables and 3-Space

- A function of several variables is, as the name indicates, a function that takes in several variables and outputs a value associated to the inputs.

   ○ <u>Example</u>: The function $f(x, y) = x + y$ takes in two values x and y and outputs their sum $x + y$.

   ○ <u>Example</u>: The function $g(x, y, z) = x\, y\, z^2$ takes in three values x, y, and z, and outputs the product $x\, y\, z^2$.

   ○ <u>Example</u>: The function $d(l, w) = \sqrt{l^2 + w^2}$ gives the length of a diagonal of a rectangle as a function of the rectangle's length l and its width w.

   ○ <u>Example</u>: The function $V(r, h) = \dfrac{1}{3}\pi r^2 h$ gives the volume of a (right circular) cone as a function of its base radius r and its height h.

- As with functions of one variable, a function of several variables has a <u>domain</u> and a <u>range</u>: the domain is the set of input values and the range is the set of output values.

  ○ For a function of two variables $f(x, y)$, the domain is now a subset of the 2-dimensional plane rather than a subset of the real line. Because of this, domains of functions of more than one variable can be rather complicated.

  ○ In general, unless specified, the domain of a function is the largest possible set of inputs for which the definition of the function makes sense. We generally adopt the conventions that square roots of negative real numbers are not allowed, nor is division by zero.

  ○ <u>Example</u>: For $f(x, y) = \sqrt{x} + \sqrt{y}$, the domain is the first quadrant of the xy-plane, defined by the inequalities $x \geq 0$ and $y \geq 0$.

  ○ <u>Example</u>: For $f(x, y) = \sqrt{x^2 + y^2 - 1}$, the domain is the set of points in the xy-plane which satisfy $x^2 + y^2 \geq 1$: this describes all the points of the plane except for those lying strictly inside the unit circle.

  ○ <u>Example</u>: For $f(x, y) = \dfrac{1}{x - y}$, the domain is the set of points in the xy-plane which satisfy $x - y \neq 0$: this describes all points in the plane except those on the line $y = x$.

- Points in <u>3-space</u> are represented by a triplet of numbers (x, y, z). The new coordinate z represents height above the xy-plane.

  ○ From an algebraic standpoint, 3-space behaves quite similarly to 2-dimensional space.

  ○ For example, in 3-space, we can measure the distance between any two points. By a suitable pair of applications of the Pythagorean Theorem, we can compute that the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

1.1.1 Graphing Functions of 2 Variables: Surfaces and Level Sets

- There are two primary ways to visualize a function $f(x, y)$ of two variables.

- The first way is to plot the points (x, y, z) in 3-dimensional space satisfying $z = f(x, y)$.

  ○ At the point (x, y) in the plane, the graph has the height $z = f(x, y)$; so we see that as (x, y) varies through the plane, the function $z = f(x, y)$ will trace out a surface, called the <u>graph</u> of $f(x, y)$.

- Some example of simple graphs $z = f(x, y)$ are given below.

  ○ <u>Example</u>: The graph $z = 0$ is the xy-plane.

  ○ <u>Example</u>: The graph $z = x + y$ is a tilted plane. More generally, an equation of the form $ax + by + cz = d$ for constants a, b, c, d (with not all of a, b, c zero) will give a plane, as we will discuss later.
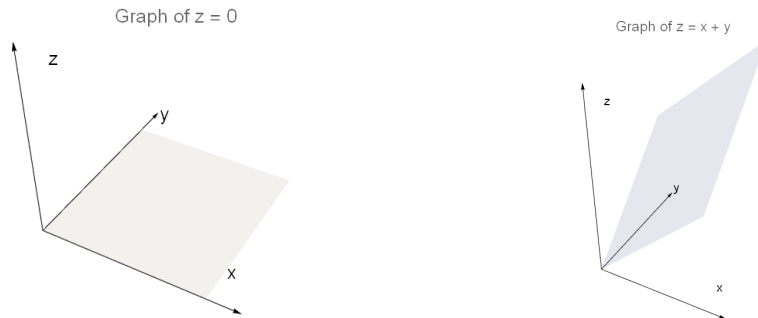

- <u>Example</u>: The graph $z = x^2 + y^2$ is a <u>paraboloid</u> (i.e., a parabolic dish).

- <u>Example</u>: The graph $z = \sqrt{x^2 + y^2}$ is a right circular <u>cone</u> opening upward, with vertex at the origin.

- Example: The graph of $z = x^2 - y^2$ is a hyperbolic paraboloid, also called a saddle surface.
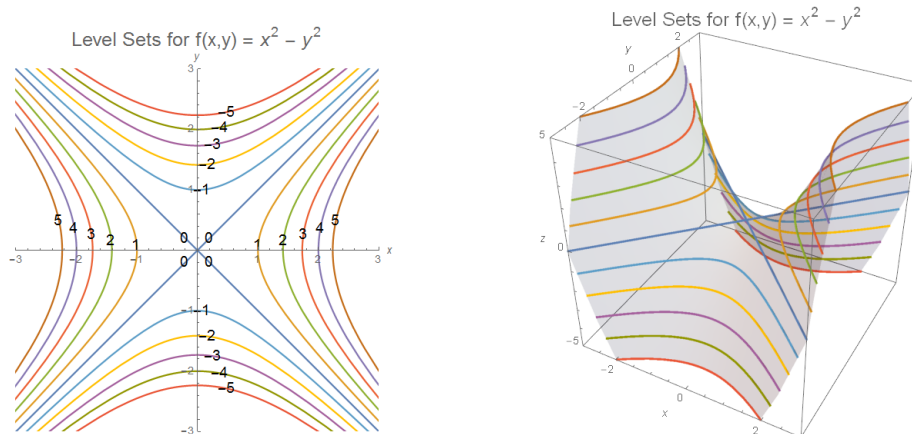
Graph of $z = x^2 + y^2$

Graph of $z = \sqrt{x^2 + y^2}$

Graph of $z = x^2 - y^2$
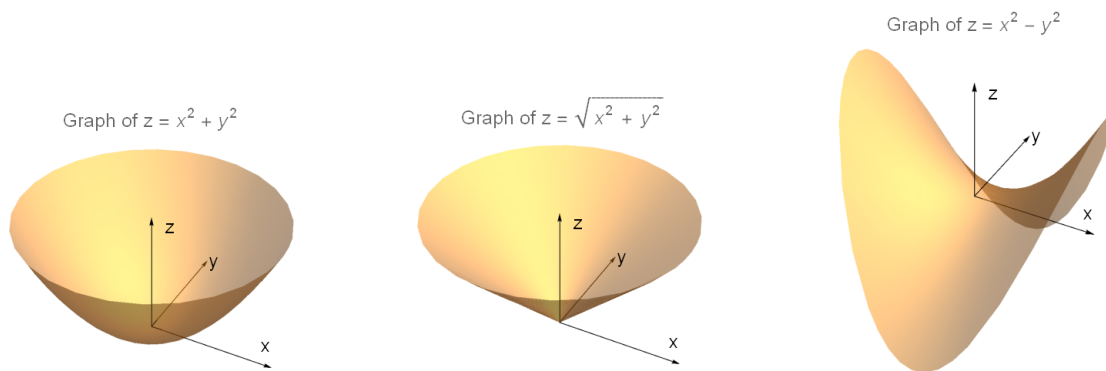
- Example: The graph $z = x^2$ is a parabolic cylinder.

- Example: The graph $z = x^3 - 3xy^2$ is sometimes called the "monkey saddle", as it has three depressions rather than the two for the regular saddle (one for each leg, and one for the tail).

- Example: The graph $z = e^{3 - \sqrt{x^2+y^2}/12} \cos\left(\sqrt{x^2 + y^2}\right)$ produces a surface that looks like ripples in a pool of water.

Graph of $z = x^2$

Graph of $z = x^3 - 3xy^2$

Graph of $z = e^{3 - \sqrt{x^2+y^2}/12} \cos(\sqrt{x^2 + y^2})$
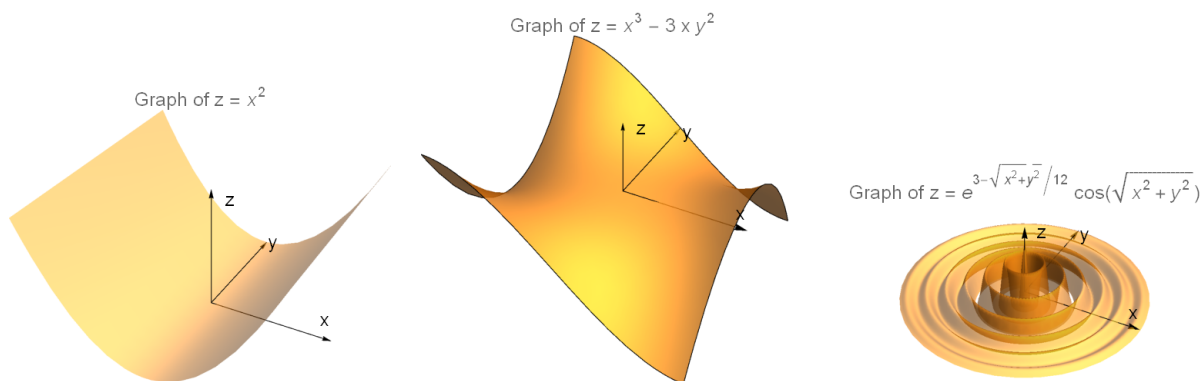
- The second way to visualize a function $f(x, y)$ is to plot the points (x, y) in the plane on the level sets $f(x, y) = c$ for particular values of c, as implicit curves.

  ○ For a given function $f(x, y)$ and a particular value of c, the points (x, y) satisfying $f(x, y) = c$ are called a level set of f.

  ○ For a function of two variables these sets will generally be curves, so they are also sometimes called level curves.

  ○ Level sets are obtained by intersecting the horizontal plane $z = c$ with the graph of the surface $z = f(x, y)$.

  ○ If we graph many of these level curves together on the same axes, we will obtain a "topographical map" of the function $f(x, y)$.

- Example: The level sets for the function $f(x, y) = x^2 + y^2$ are circles in the plane.

  ○ More specifically, the level set $x^2 + y^2 = c$ (for $c > 0$) is a circle with radius $\sqrt{c}$ centered at $(0, 0)$.

  ○ For $c = 0$ the level set is just the single point $(0, 0)$, and for $c < 0$ the level sets do not contain any points at all.

  ○ Below on the left are plotted the level curves in the plane for $c = 1, 2, 3, \ldots, 9$, while on the right the level sets have been drawn on the graph of the surface $z = f(x, y)$ itself, to illustrate how the level curves arise by intersecting the surface $z = f(x, y)$ with the various horizontal planes.

3

Level Sets for f(x,y) = $x^2 + y^2$
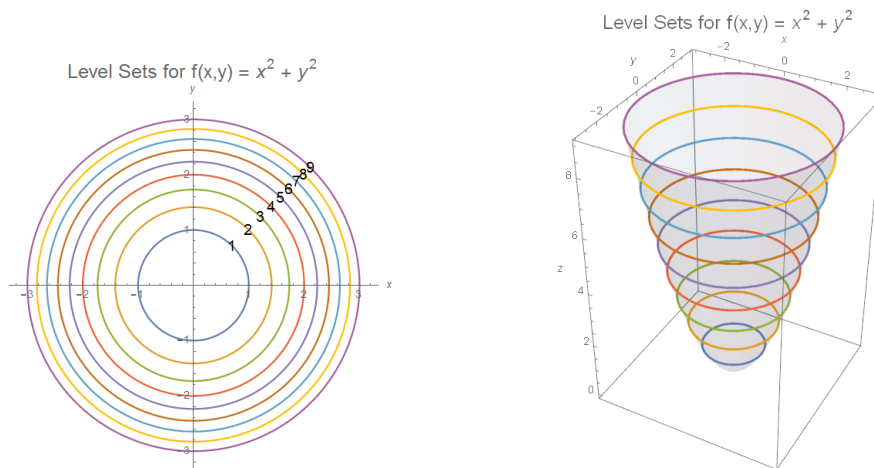
- **Example:** The level sets for the function $f(x,y) = x^2 - y^2$ are hyperbolas.

    ○ More specifically, the level set $x^2 - y^2 = c$ (for $c \neq 0$) are hyperbolas (each one consists of two pieces), while for $c = 0$ the level set is the union of the two lines $y = x$ and $y = -x$.

    ○ Below on the left are plotted the level curves in the plane for $c = -5, -4, -3, \ldots, 4, 5$, while on the right the level sets have been drawn on the graph of the surface $z = f(x,y)$ itself.

    ○ As we noted earlier, the graph of $z = x^2 - y^2$ is called a hyperbolic paraboloid: it has this name because its horizontal cross-sections (corresponding to the level sets) are hyperbolas while its vertical cross-sections are parabolas.

- **Example:** Below are plotted the level curves for the function $f(x,y) = (x^2 - y^2)^2 e^{-x^2 - y^2}$ for $c = 0.04, 0.14, \ldots 0.54$, along with the 3-dimensional plot of $z = f(x,y)$:
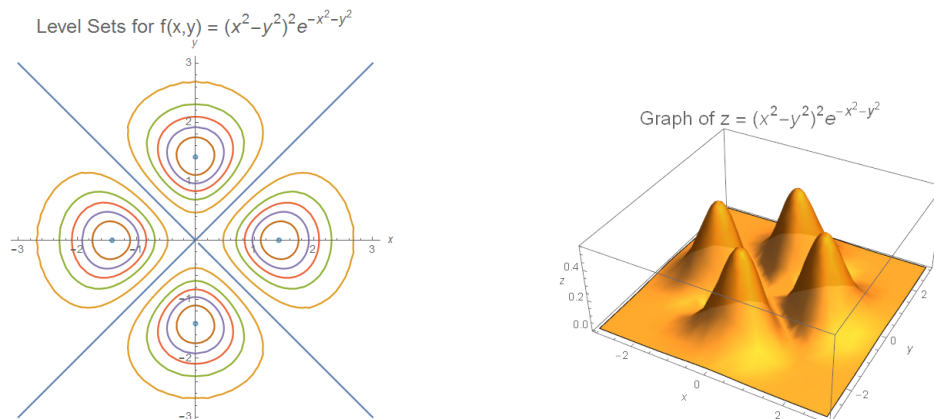
○ As one can see from comparing the plots, the level curves indicate where the function is changing in value: many level curves grouped closely together indicates that the function is changing quickly so that the function's graph will be "steep", while having level curves grouped far apart means that the function's graph will be fairly flat.

1.1.2 Graphing Functions of 3 Variables: Level Surfaces

- We can also represent some information graphically for functions $f(x, y, z)$ of three variables. Unfortunately, we cannot really produce proper graphs of these functions, since plotting a graph $w = f(x, y, z)$ would require drawing a 4-dimensional picture.

- However, we can still talk about level sets of functions of 3 variables – these are the points (x, y, z) satisfying a relation $f(x, y, z) = c$: these will (in general) give rise to <u>level surfaces</u> in 3-dimensional space.

  ○ Any graph $z = g(x, y)$ is an example of a level surface for the function $f(x, y, z) = g(x, y) - z$: the points (x, y, z) with $z = g(x, y)$ are the same as those with $f(x, y, z) = 0$.

- <u>Example</u>: The level surfaces for the function $f(x, y, z) = x^2 + y^2 + z^2$ are spheres centered at the origin.

  ○ Explicitly, by the distance formula, the points satisfying $x^2 + y^2 + z^2 = c$ (for $c > 0$) are precisely those which are at a distance of $\sqrt{c}$ from the origin, which is merely another way of describing the sphere of radius $\sqrt{c}$ centered at $(0, 0, 0)$.

  ○ Here is a plot of the sphere $x^2 + y^2 + z^2 = 1$ (the others are geometrically similar, just with a different radius).

Graph of $x^2 + y^2 + z^2 = 1$
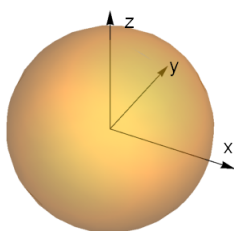
Graph of $x^2 + z^2 = 1$
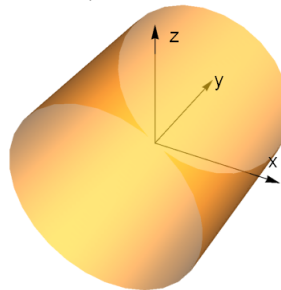
- <u>Example</u>: The level surfaces for the function $f(x, y, z) = x^2 + z^2$ are (right circular) cylinders running along the y-axis.

  ○ Explicitly, from the distance formula, points satisfying $x^2 + z^2 = c$ (for $c > 0$) are those which are at a distance of $\sqrt{c}$ from the y-axis, which is another way of describing a right circular cylinder oriented along the y-axis. A graph of the cylinder $x^2 + z^2 = 1$ is above (the others look geometrically similar, just with a different radius).

- <u>Example</u>: The level surfaces for the function $f(x, y, z) = x^2 + y^2 - z^2$ have a variety of different shapes depending on the value of the function.

  ○ The set of points satisfying $x^2 + y^2 - z^2 = c$ for $c > 0$ forms a surface called a hyperboloid of one sheet, so named because the shape is a hyperbola in two cross-sections and an ellipse in the third, and the graph is connected ("one sheet").

  ○ The set of points satisfying $x^2 + y^2 - z^2 = 0$ forms a (right circular) double cone whose axis is the z-axis and whose center point at the origin $(0, 0, 0)$.

  ○ The set of points satisfying $x^2 + y^2 - z^2 = c$ for $c < 0$ forms a surface called a hyperboloid of two sheets, so named because the shape is a hyperbola in two cross-sections and an ellipse in the third, and the graph consists of two pieces ("two sheets").

○ Here are plots of the level surfaces for $c = 1$, $c = 0$, and $c = -1$ respectively:

Graph of $x^2 + y^2 - z^2 = 1$ Graph of $x^2 + y^2 - z^2 = 0$ Graph of $x^2 + y^2 - z^2 = -1$
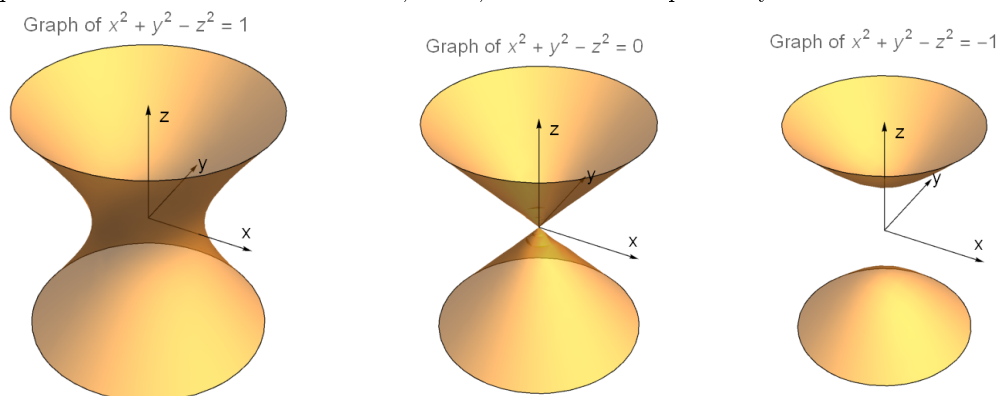
- As a final remark, we will mention that many of the surfaces we have just discussed are examples of quadric surfaces: level sets of quadratic polynomials in x, y, z.

    ○ Quadric surfaces are the 3-dimensional analogue of the famous conic sections (the ellipse, parabola, and hyperbola) often studied in precalculus courses.
    ○ The general equation for a quadric surface is $Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 + Gx + Hy + Iz = J$, for constants $A, B, C, D, E, F, G, H, I, J$.
    ○ By changing variables, one can eliminate the cross terms, and then complete the squares to eliminate other linear terms.
    ○ Other than degenerate surfaces (such as the single point $x^2 + y^2 + z^2 = 0$, the plane $x = 0$, or the pair of planes $xy = 0$), there are nine interesting types of quadric surfaces: the ellipsoid, the hyperboloid of one sheet, the hyperboloid of two sheets, the paraboloid, the hyperbolic paraboloid, the elliptical cone, and the elliptical cylinder, parabolic cylinder, and hyperbolic cylinder.

- Many of the quadric surfaces have interesting or useful physical properties, and for this reason they frequently show up in engineering and industry.

    ○ Spheres, cones, and cylinders are all very naturally useful surfaces, and are studied in elementary geometry.
    ○ The paraboloid is used as the shape of a satellite dish, since parallel rays coming from a long distance will all reflect into its focus.
    ○ The hyperboloid of one sheet and the hyperbolic paraboloid are examples of what are called ruled surfaces: through each point on the surface pass two lines which are contained in the surface.
    ○ Ruled surfaces can therefore be physically constructed using materials which are not curved.
    ○ The hyperboloid of one sheet, in particular, is a common design for cooling towers, while the saddle surface has been used as the roof design for certain sports arenas.

1.2   Vectors, Dot and Cross Products, Lines and Planes

- In this section we introduce vectors and a pair of important vector operations, the dot and cross products, which will allow us to study the equations of lines and planes in 3-space.

1.2.1   Vectors and Vector Operations

- A vector is a quantity which has both a magnitude and a direction.

    ○ This is in contrast to a scalar, which carries only a magnitude.

- We denote the n-dimensional vector from the origin to the point $(a_1, a_2, \cdots, a_n)$ as $\mathbf{v} = \langle a_1, a_2, \cdots, a_n \rangle$, where the a_i are scalars.

○ We use the angle brackets $\langle \cdot \rangle$ rather than parentheses $(\cdot)$ so as to emphasize the difference between a vector and the coordinates of a point in space. We will, however, view coordinates of vectors and coordinates of points as essentially interchangeable.

○ We also write vectors in boldface ($\mathbf{v}$, not v), so that we can tell them apart from scalars. When writing by hand, it is hard to differentiate boldface, so the notation $\vec{v}$ is also sometimes used.

- The typical way to think of vectors is as directed line segments: the length of the line segment gives the magnitude of the vector, and the direction the segment is pointing gives the direction of the vector.

  ○ As a warning, we remark that vectors are a little bit different from directed line segments, because we don't care where a vector starts: we only care about the difference between the starting and ending positions.

  ○ Thus, the directed segment whose start is $(0,0)$ and end is $(1,1)$ and the segment starting at $(1,1)$ and ending at $(2,2)$ represent *the same vector* $\langle 1,1 \rangle$. This distinction is rarely necessary in most applications, but it is a useful thing to keep in mind when visualizing vectors, since we can view any vector as having any arbitrary starting point we choose.

- We can add vectors (provided they are of the same dimension!) in the obvious way, one component at a time:

- <u>Definition</u>: If $\mathbf{v} = \langle a_1, \cdots, a_n \rangle$ and $\mathbf{w} = \langle b_1, \cdots, b_n \rangle$ of the same length, then their <u>sum</u> is $\mathbf{v} + \mathbf{w} = \langle a_1 + b_1, \cdots, a_n + b_n \rangle$.

  ○ We can justify this using our geometric idea of what a vector does: $\mathbf{v}$ moves us from the origin to the point $(a_1, \cdots, a_n)$. Then $\mathbf{w}$ tells us to add $\langle b_1, \cdots, b_n \rangle$ to the coordinates of our current position, and so $\mathbf{w}$ moves us from $(a_1, \cdots, a_n)$ to $(a_1 + b_1, \cdots, a_n + b_n)$. So the net result is that the sum vector $\mathbf{v} + \mathbf{w}$ moves us from the origin to $(a_1 + b_1, \cdots, a_n + b_n)$, meaning that it is just the vector $\langle a_1 + b_1, \cdots, a_n + b_n \rangle$.

  ○ Geometrically, we can think of vector addition using a parallelogram whose pairs of parallel sides are $\mathbf{v}$ and $\mathbf{w}$ and whose diagonal is $\mathbf{v} + \mathbf{w}$:

Parallelogram for Vector Addition
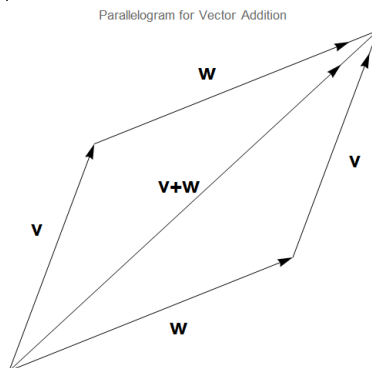
- We can also 'scale' a vector by a scalar, one component at a time:

- <u>Definition</u>: If r is a scalar and $\mathbf{v} = \langle a_1, \ldots, a_n \rangle$ is a vector, then the scalar multiple $r\mathbf{v}$ is defined as $r\,\mathbf{v} = \langle ra_1, \cdots, ra_n \rangle$.

  ○ Again, we can justify this by our geometric idea of what a vector does: if $\mathbf{v}$ moves us some amount in a direction, then $\dfrac{1}{2}\mathbf{v}$ should move us half as far in that direction.

  ○ Analogously, $2\mathbf{v}$ should move us twice as far in that direction, while $-\mathbf{v}$ should move us exactly as far, but in the opposite direction.

- <u>Example</u>: If $\mathbf{v} = \langle -1, 2, 2 \rangle$ and $\mathbf{w} = \langle 3, 0, -4 \rangle$, find $2\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, and $\mathbf{v} - 2\mathbf{w}$.

  ○ We have $2\mathbf{w} = \langle 2 \cdot 3, 2 \cdot 0, 2 \cdot (-4) \rangle = \boxed{\langle 6, 0, -8 \rangle}$, $\mathbf{v} + \mathbf{w} = \langle -1 + 3, 2 + 0, 2 + (-4) \rangle = \boxed{\langle 2, 2, -2 \rangle}$.

  ○ Then $\mathbf{v} - 2\mathbf{w} = \langle -1, 2, 2 \rangle - \langle 6, 0, -8 \rangle = \boxed{\langle -7, 2, 10 \rangle}$.

- Arithmetic of vectors satisfies several algebraic properties that follow from the definition:

    ○ Addition of vectors is commutative ($\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$) and associative ($\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$).
    ○ There is a zero vector $\mathbf{0}$ (namely, the vector with all entries zero), and every vector $\mathbf{v}$ has an additive inverse $-\mathbf{v}$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
    ○ Scalar multiplication distributes over addition of vectors ($r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$) and scalars ($(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$).

- At this point, we will restrict ourselves to talking just about 2-dimensional space and 3-dimensional space. Our primary reason for this is that most of the immediate applications of vectors (e.g., to physics) happen in 3-dimensional space.

- It will be useful to have a way to denote the "unit coordinate" vectors of 3-dimensional space.

- <u>Definition</u>: The <u>unit coordinate vectors</u> $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$.

    ○ We can rewrite any vector in 2-space or 3-space as a linear combination of the unit coordinate vectors.
    ○ <u>Example</u>: The vector $\langle 3, 2, -5 \rangle$ is equal to $3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.
    ○ <u>Note</u>: Some authors primarily use **ijk**-notation when working with vectors. We will generally use the angle bracket notation, except in a few cases when there are useful mnemonics that are easier to remember using **ijk**-notation.

1.2.2 The Dot Product

- We now analyze lengths and angles between vectors.
- <u>Definition</u>: We define the <u>norm</u> (also called the <u>length</u> or <u>magnitude</u>) of the vector $\mathbf{v} = \langle a_1, \ldots, a_n \rangle$ as $||\mathbf{v}|| = \sqrt{(a_1)^2 + \cdots + (a_n)^2}$.

    ○ This is just an application of the distance formula: the norm of the vector $\langle a_1, \ldots, a_n \rangle$ is just the length of the line segment joining the origin $(0, \ldots, 0)$ to the point $(a_1, \ldots, a_n)$.
    ○ <u>Example</u>: For $\mathbf{v} = \langle -1, 2, 2 \rangle$ and $\mathbf{w} = \langle 3, 0, -4 \rangle$, we have $||\mathbf{v}|| = \sqrt{(-1)^2 + 2^2 + 2^2} = \boxed{3}$, and $||\mathbf{w}|| = \sqrt{3^2 + 0^2 + (-4)^2} = \boxed{5}$.
    ○ If r is a scalar, we can see immediately from the definition that $||r\,\mathbf{v}|| = |r|\,||\mathbf{v}||$, since we can just factor out a $\sqrt{r^2} = |r|$ from each term under the square root.

- From any nonzero vector we can construct a unit vector (that is, a vector of length 1) in the same direction of $\mathbf{v}$ just by scaling $\mathbf{v}$ by 1 over its length.

- <u>Definition</u>: If $\mathbf{v}$ is a nonzero vector, the vector $\mathbf{u} = \dfrac{\mathbf{v}}{||\mathbf{v}||}$, called the <u>normalization</u> of $\mathbf{v}$, is a unit vector in the same direction as $\mathbf{v}$.

    ○ <u>Example</u>: For $\mathbf{v} = \langle -1, 2, 2 \rangle$, since $||\mathbf{v}|| = 3$, we see that $\mathbf{u}_1 = \boxed{\left\langle -\dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3} \right\rangle}$ is a unit vector in the same direction as $\mathbf{v}$.
    ○ <u>Example</u>: For $\mathbf{w} = \langle 3, 0, -4 \rangle$, since $||\mathbf{w}|| = 5$, we see that $\mathbf{u}_2 = \boxed{\left\langle \dfrac{3}{5}, 0, -\dfrac{4}{5} \right\rangle}$ is a unit vector in the direction of $\mathbf{w}$.

- But another thing we might want to know about two vectors is the angle θ between them. This motivates the definition of the dot product:

- <u>Definition</u>: The <u>dot product</u> of two vectors $\mathbf{v}_1 = \langle a_1, \ldots, a_n \rangle$ and $\mathbf{v}_2 = \langle b_1, \ldots, b_n \rangle$ is defined to be the scalar $\mathbf{v}_1 \cdot \mathbf{v}_2 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

- Note that the dot product of two vectors is a scalar, *not* a vector! For this reason, the dot product is sometimes called the <u>scalar product</u>.

- <u>Example</u>: The dot product $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$ is $(1)(3) + (2)(4) = \boxed{11}$.

- <u>Example</u>: The dot product $\langle -1, 2, 2 \rangle \cdot \langle 3, 0, -4 \rangle$ is $(-1)(3) + (2)(0) + (2)(-4) = \boxed{-11}$.

- The dot product possesses several numerous properties reminiscent of standard multiplication.

- <u>Proposition</u> (Properties of Dot Products): For any vectors $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$, and any scalar r, the following properties hold:

  1. The dot product distributes over addition: $(\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w} = (\mathbf{v}_1 \cdot \mathbf{w}) + (\mathbf{v}_2 \cdot \mathbf{w})$.
  2. The dot product distributes through scaling: $(r\mathbf{v}) \cdot \mathbf{w} = r(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (r\mathbf{w})$.
  3. The dot product is commutative: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
  4. The dot product of a vector with itself is the square of the norm: $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$. (In particular, $\mathbf{v} \cdot \mathbf{v} \geq 0$ for all vectors $\mathbf{v}$.)

     - <u>Proofs</u>: Each of these properties is a simple algebraic calculation from the definition of the dot product.

- There is a nice relation between the dot product and the angle between two vectors:

- <u>Theorem</u> (Dot Product): For vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ forming an angle θ between them, we have $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \, \cos(\theta)$.

  - <u>Proof</u>: Apply the Law of Cosines in the triangle formed by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_2 - \mathbf{v}_1$: this yields $||\mathbf{v}_2 - \mathbf{v}_1||^2 = ||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2 \, ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \cos(\theta)$.

  - Since $||\mathbf{w}||^2 = \mathbf{w} \cdot \mathbf{w}$ for any vector $\mathbf{v}$, we can then convert the statement above to one involving dot products, and then apply the various properties of dot products:

$$
\begin{aligned}
||\mathbf{v}_2 - \mathbf{v}_1||^2 &= (\mathbf{v}_2 - \mathbf{v}_1) \cdot (\mathbf{v}_2 - \mathbf{v}_1) \\
&= (\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2) - (\mathbf{v}_2 \cdot \mathbf{v}_1) + (\mathbf{v}_1 \cdot \mathbf{v}_1) \\
&= ||\mathbf{v}_2||^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2) + ||\mathbf{v}_1||^2 .
\end{aligned}
$$

  - Then, by comparing the expression above to the Law of Cosines expression and cancelling terms, we are left with $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \, \cos(\theta)$, as desired.

- Using the Dot Product Theorem, we can compute the angle between two vectors:

- <u>Example</u>: Compute the angle between the vectors $\mathbf{v} = \langle 2, 1, \sqrt{3} \rangle$ and $\mathbf{w} = \langle 0, \sqrt{3}, 1 \rangle$.

  - We compute $\mathbf{v} \cdot \mathbf{w} = (2)(0) + (1)(\sqrt{3}) + (\sqrt{3})(1) = 2\sqrt{3}$, and $||\mathbf{v}|| = \sqrt{2^2 + 1^2 + (\sqrt{3})^2} = 2\sqrt{3}$ and $||\mathbf{w}|| = \sqrt{(\sqrt{3})^2 + 0^2 + 1^2} = 2$.

  - Then by the Dot Product Theorem, the angle θ between the vectors satisfies $2\sqrt{3} = 2 \cdot 2\sqrt{3} \cdot \cos(\theta)$, meaning that $\theta = \cos^{-1}\left(\dfrac{1}{2}\right) = \boxed{\dfrac{\pi}{3}}$.

- <u>Example</u>: Compute the angle between the vectors $\mathbf{v} = \langle 2, 2, -1 \rangle$ and $\mathbf{w} = \langle 3, 4, 0 \rangle$.

  - We compute $\mathbf{v} \cdot \mathbf{w} = (2)(3) + (2)(4) + (-1)(0) = 14$, and $||\mathbf{v}|| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$ and $||\mathbf{w}|| = \sqrt{3^2 + 0^2 + (-4)^2} = 5$.

  - Then by the Dot Product Theorem, the angle θ between the vectors satisfies $14 = 3 \cdot 5 \cdot \cos(\theta)$, so $\theta = \boxed{\cos^{-1}\left(\dfrac{14}{15}\right)}$.

- Using the Dot Product Theorem, we can see that the sign and magnitude of the dot product is (roughly) measuring whether the vectors are pointing in the same direction:

  ○ If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors, then both $||\mathbf{v}_1||$ and $||\mathbf{v}_2||$ are positive, so by the theorem above, the value $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \cos(\theta)$ will have the same sign as $\cos(\theta)$.

  ○ If $0 \leq \theta < \dfrac{\pi}{2}$, the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be positive. Furthermore, the smaller θ is, the larger the value of $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be. Thus, a large positive value for the dot product indicates that the vectors are pointing in roughly the same direction.

  ○ Inversely, if $\dfrac{\pi}{2} < \theta \leq \pi$, the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be negative, and the larger θ is, the larger negative $\mathbf{v}_1 \cdot \mathbf{v}_2$ will be. Thus, a large negative value for the dot product indicates that the vectors are pointing in roughly opposite directions.

- We have a special name for the case where the angle between two vectors is $\pi/2$:

- <u>Definition</u>: We say two vectors are <u>orthogonal</u> if their dot product is zero.

  ○ From the Dot Product Theorem, since $\cos(\pi/2) = 0$, we see that two nonzero vectors are orthogonal if the angle between them is $\pi/2$, which is to say, if they are perpendicular.

  ○ <u>Example</u>: The vectors $\langle 2, -1, 4 \rangle$ and $\langle 3, 2, -1 \rangle$ are orthogonal, since their dot product is $(2)(3)+(-1)(2)+(4)(-1) = 0$.

  ○ <u>Remark</u>: Since the dot product of the zero vector with any vector is zero, by our definition above, the zero vector is orthogonal to every vector.

- Another basic question about vectors is: given a vector $\mathbf{v}$ and another vector $\mathbf{w}$, how much of $\mathbf{w}$ is in the direction of $\mathbf{v}$, and how much of $\mathbf{w}$ is orthogonal to $\mathbf{v}$? (This is a problem that often arises in Newtonian physics: one often needs to separate the vector representing a force into orthogonal components.)

  ○ In other words: we want to write $\mathbf{w} = a\mathbf{v} + \mathbf{y}$, where $\mathbf{y}$ is orthogonal to $\mathbf{v}$. The goal is to determine the value of the scalar a.

  ○ Since $\mathbf{y} = \mathbf{w} - a\mathbf{v}$, and $\mathbf{y}$ is orthogonal to $\mathbf{v}$, taking the dot product with $\mathbf{v}$ gives $0 = \mathbf{y} \cdot \mathbf{v} = (\mathbf{w} - a\mathbf{v}) \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v} - a(\mathbf{v} \cdot \mathbf{v})$.

  ○ Solving for a gives $a = \dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$.

- <u>Definition</u>: The <u>vector projection</u> of $\mathbf{w}$ onto the nonzero vector $\mathbf{v}$ is the vector $\mathrm{Proj}_{\mathbf{v}}(\mathbf{w}) = \left(\dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v}$.

  ○ The vector projection of $\mathbf{w}$ onto $\mathbf{v}$ gives the "piece" of $\mathbf{w}$ in the direction of $\mathbf{v}$, while the remaining piece $\mathbf{w} - \mathrm{Proj}_{\mathbf{v}}(\mathbf{w})$ is orthogonal to $\mathbf{v}$.

- <u>Example</u>: Find the vector projection of $\mathbf{w} = \langle 3, 4, -2 \rangle$ onto the coordinate vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

  ○ We have $\mathrm{Proj}_{\mathbf{i}}(\mathbf{w}) = \dfrac{\mathbf{w} \cdot \mathbf{i}}{\mathbf{i} \cdot \mathbf{i}} \mathbf{i} = \dfrac{3}{1} \mathbf{i} = 3\mathbf{i} = \langle 3, 0, 0 \rangle$.

  ○ Next, $\mathrm{Proj}_{\mathbf{j}}(\mathbf{w}) = \dfrac{\mathbf{w} \cdot \mathbf{j}}{\mathbf{j} \cdot \mathbf{j}} \mathbf{j} = \dfrac{4}{1} \mathbf{j} = 4\mathbf{j} = \langle 0, 4, 0 \rangle$.

  ○ Finally, $\mathrm{Proj}_{\mathbf{k}}(\mathbf{w}) = \dfrac{\mathbf{w} \cdot \mathbf{k}}{\mathbf{k} \cdot \mathbf{k}} \mathbf{k} = \dfrac{-2}{1} \mathbf{k} = -2\mathbf{k} = \langle 0, 0, -2 \rangle$.

  ○ These results are consistent with our intuition about what the vector projection should be: for example, the component of $\langle 3, 4, -2 \rangle$ in the direction of $\langle 1, 0, 0 \rangle$ should logically be $\langle 3, 0, 0 \rangle$, and this is indeed what we got.

- <u>Example</u>: Find the vector projection of $\mathbf{w} = \langle 6, -3, 0 \rangle$ onto the vector $\mathbf{v} = \langle 2, -2, -1 \rangle$.

  ○ We have $\mathrm{Proj}_{\mathbf{v}}(\mathbf{w}) = \left(\dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \dfrac{(6)(2) + (-3)(-2) + (0)(-1)}{2^2 + (-2)^2 + (-1)^2} \mathbf{v} = \dfrac{18}{9} \mathbf{v} = 2\mathbf{v} = \boxed{\langle 4, -4, -2 \rangle}$.

  ○ As a sanity check, note that $\mathbf{w} - \mathrm{Proj}_{\mathbf{v}}(\mathbf{w}) = \langle 2, 1, 2 \rangle$, and indeed, this vector is orthogonal to $\mathbf{v}$, because $\langle 2, 1, 2 \rangle \cdot \langle 2, -2, -1 \rangle = (2)(2) + (1)(-2) + (2)(-1) = 0$.

1.2.3 The Cross Product

- In addition to the dot product, we have another type of product defined for vectors in 3-space, called the cross product:

- Definition: The cross product of $\mathbf{v}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v}_2 = \langle x_2, y_2, z_2 \rangle$ is defined to be the vector $\mathbf{v}_1 \times \mathbf{v}_2 = \langle y_1 z_2 - y_2 z_1, \ z_1 x_2 - z_2 x_1, \ x_1 y_2 - x_2 y_2 \rangle$. It is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

  - Important Note: The cross product is only defined for vectors with 3 components, and outputs another vector with 3 components. In contrast, the dot product is defined for vectors of any length, and outputs a scalar.

  - A way to remember the cross product formula (aside from memorization) is the "determinant formula"
$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}, \text{ where } \mathbf{i}, \mathbf{j}, \mathbf{k} \text{ are the standard}$$
  unit vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. (It may seem a little unusual to have vectors inside a determinant, but it works out to the correct answer.)

  - Warning: Don't forget the minus sign on the middle term in the determinant formula above!

- The fundamental property of the cross product $\mathbf{v}_1 \times \mathbf{v}_2$ is that it is orthogonal both to $\mathbf{v}_1$ and to $\mathbf{v}_2$.

  - To verify this, we can just evaluate the dot products $\mathbf{v}_1 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)$ and $\mathbf{v}_2 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)$ and check that they are both zero.

  - For example, we have $\mathbf{v}_1 \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = x_1(y_1 z_2 - y_2 z_1) + y_1(z_1 x_2 - z_2 x_1) + z_1(x_1 y_2 - x_2 y_2)$, which some algebra will confirm is equal to zero.

- Example: If $\mathbf{v}_1 = \langle 1, 1, 3 \rangle$ and $\mathbf{v}_2 = \langle 2, -1, 1 \rangle$, find $\mathbf{v}_1 \times \mathbf{v}_2$ and $\mathbf{v}_2 \times \mathbf{v}_1$ and verify that $\mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

  - First, $\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{k} = \boxed{\langle 4, 5, -3 \rangle}$.

  - Also, $\mathbf{v}_2 \times \mathbf{v}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \mathbf{k} = \boxed{\langle -4, -5, 3 \rangle}$.

  - To check: we have $\mathbf{v}_1 \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = \langle 1, 1, 3 \rangle \cdot \langle 4, 5, -3 \rangle = 4 + 5 - 9 = 0$, and also $\mathbf{v}_2 \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = \langle 2, -1, 1 \rangle \cdot \langle 4, 5, -3 \rangle = 8 - 5 - 3 = 0$.

- Here are a few other algebraic properties of the cross product:

- Proposition (Properties of Cross Products): For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$, and any scalar r, the following properties hold:

  1. The cross product of two vectors is orthogonal to both vectors: $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.
  2. The cross product distributes over addition: $(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{w} = (\mathbf{v}_1 \times \mathbf{w}) + (\mathbf{v}_2 \times \mathbf{w})$.
  3. The cross product distributes through scaling: $(r\mathbf{v}) \times \mathbf{w} = r(\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (r\mathbf{w})$.
  4. The dot product is anticommutative: $\mathbf{v} \times \mathbf{w} = -(\mathbf{v} \times \mathbf{w})$. In particular, $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ for any $\mathbf{v}$.

     - Proofs: Each of these properties is a simple algebraic calculation from the definition of the cross product.
     - We will also mention that the cross product satisfies a relation somewhat akin to associativity, the so-called "Jacobi identity": $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.

- Like with the dot product, there is a relationship involving the cross product and the angle between two vectors:

- Theorem (Cross Product): If θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, then $||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sin(\theta) = A$, where A is the area of the parallelogram formed by $\mathbf{v}_1$ and $\mathbf{v}_2$.

- ○ <u>Proof</u>: We just need to show that $||\mathbf{v}_1 \times \mathbf{v}_2||^2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = ||\mathbf{v}_1||^2 \, ||\mathbf{v}_2||^2$, because we know that $\mathbf{v}_1 \cdot \mathbf{v}_2 = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \cos(\theta)$ from the Dot Product Theorem.
- ○ To check this, we simply multiply it out, which reduces to showing that $(y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_2)^2 + (x_1 x_2 + y_1 y_2 + z_1 z_2)^2$ is equal to $\left[(x_1)^2 + (y_1)^2 + (z_1)^2\right] \cdot \left[(x_2)^2 + (y_2)^2 + (z_2)^2\right]$.
- ○ When we expand the first quantity, we get each of the 9 possible square terms $(\square_1 \triangle_2)^2$ where $\square$ and $\triangle$ are each one of x, y, or z, and the "cross" terms like $2x_1 x_2 y_1 y_2$ will all cancel out. We get exactly the same sum of 9 square terms when we expand the second quantity: thus, they are equal.
- ○ Therefore, $||\mathbf{v}_1 \times \mathbf{v}_2||^2 + (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 = ||\mathbf{v}_1||^2 \, ||\mathbf{v}_2||^2$, and so $||\mathbf{v}_1 \times \mathbf{v}_2|| = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sqrt{1 - \cos^2(\theta)} = ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sin(\theta)$ since the norm of the cross product is nonnegative as is $\sin(\theta)$.
- ○ For the statement about the area, it is a standard fact of geometry that the area of the triangle with sides $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\frac{1}{2} ||\mathbf{v}_1|| \, ||\mathbf{v}_2|| \sin(\theta)$. The parallelogram's area is twice this.
- ○ <u>Remark</u>: This area property is one reason we chose the definition we did for the cross product.

- **Example**: Find the area of the triangle whose vertices are the points $A(1, -1, 2)$, $B(2, -3, 1)$, and $C(2, 2, 2)$.

  - ○ By the Cross Product Theorem, the area of the triangle is $A = \frac{1}{2} ||\mathbf{v} \times \mathbf{w}||$, where $\mathbf{v}$ and $\mathbf{w}$ are vectors representing two sides of the triangle.
  - ○ We can take $\mathbf{v} = B - A = \langle 1, -2, -1 \rangle$ and $\mathbf{w} = C - B = \langle 0, 5, 1 \rangle$.
  - ○ Then $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ 0 & 5 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ 5 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} \mathbf{k} = \langle 3, -1, 5 \rangle$.
  - ○ Hence the area of the triangle is $\frac{1}{2} ||\langle 3, -1, 5 \rangle|| = \frac{1}{2} \sqrt{3^2 + (-1)^2 + 5^2} = \boxed{\dfrac{\sqrt{35}}{2}}$.

1.2.4 Lines and Planes in 3-Space

- Among the many useful applications of vectors is that they can be used to give simple descriptions of lines and planes in 3-space.

  - ○ In the 2-dimensional plane, the general equation of a line is $ax + by = d$ for some constants a, b, d (with a and b not both zero). The 3-dimensional version of this would be $ax + by + cz = d$ (with a, b, c not all zero), which will describe a plane.
  - ○ To describe lines in 3-space, we instead need to describe them as parametric curves, in the form $x = x(t)$, $y = y(t)$, $z = z(t)$ for some functions $x(t)$, $y(t)$, $z(t)$. As t varies, the set of points $(x(t), y(t), z(t))$ will form a curve: the goal is to choose the functions so that the curve is a line.

- <u>Proposition</u> (Parametrization of a Line): Given distinct points $P_1 = \langle x_1, y_1, z_1 \rangle$ and $P_2 = \langle x_2, y_2, z_2 \rangle$, the points $\langle x, y, z \rangle$ on the line l through P_1 and P_2 are given parametrically by $\langle x, y, z \rangle = P_1 + t \, \mathbf{v}$, as t varies through the real numbers and $\mathbf{v} = P_2 - P_1 = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. The equation can be written explicitly as $\langle x, y, z \rangle = \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1), z_1 + t(z_2 - z_1) \rangle$.

  - ○ <u>Proof</u>: There is a unique line between two points, by the axioms of geometry, so we need only verify that $l : \langle x, y, z \rangle = P_1 + t \, \mathbf{v}$ is a line and that it goes through P_1 and P_2.
  - ○ The parametric equation for l explicitly tells us that $x = x_1 + t(x_2 - x_1)$, $y = y_1 + t(y_2 - y_1)$, and $z = z_1 + t(z_2 - z_1)$, and these are all linear equations: thus, l is a line.
  - ○ Setting $t = 0$ yields P_1, while setting $t = 1$ yields $P_1 + (P_2 - P_1) = P_2$. Thus, l passes through both P_1 and P_2, so it is the desired line.
  - ○ <u>Note</u>: This procedure works to find the parametrization of a line in a space of any dimension, not just 3-space.
  - ○ <u>Remark</u>: We call the vector $\mathbf{v} = P_2 - P_1$ the <u>direction vector</u> for the line l: it tells us in which direction the line is pointing. The term P_1 in the sum $P_1 + t \, \mathbf{v}$ specifies which particular line we want, of all possible lines in that direction.

- <u>Example</u>: Find a parametrization of the line through the points $(1, 2, 3)$ and $(-1, 2, -1)$.

  - First, we find the direction vector: we get $\mathbf{v} = \langle -1, 2, -1 \rangle - \langle 1, 2, 3 \rangle = \langle -2, 0, -4 \rangle$.
  - Then the line is given parametrically by $\boxed{\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + t \langle -2, 0, -4 \rangle = \langle 1 - 2t, \, 2, \, 3 - 4t \rangle}$.
  - More explicitly, this line is described by $x = 1 - 2t$, $y = 2$, $z = 3 - 4t$. As t ranges through the real numbers, the points (x, y, z) will range along the desired line (so for example, setting $t = -2$ yields another point $\langle 5, 2, 11 \rangle$ on the line.)

- Next, we tackle descriptions of planes.

- <u>Proposition</u> (Planes and Normal Vectors): The plane defined by $ax + by + cz = d$ is orthogonal to its <u>normal vector</u> $\mathbf{n} = \langle a, b, c \rangle$. In other words, every line lying in this plane is orthogonal to $\langle a, b, c \rangle$. Conversely, given a nonzero vector $\mathbf{n} = \langle a, b, c \rangle$, there is a unique plane normal to that vector passing through a given point (x_0, y_0, z_0), and its equation is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

  - <u>Proof</u>: Suppose l is a line in the plane. All we need to show is that its direction vector is orthogonal to $\mathbf{n}$.
  - So suppose the direction vector is $\mathbf{v} = \mathbf{P}_2 - \mathbf{P}_1$, where both of the points $\mathbf{P}_2 = \langle x_2, y_2, z_2 \rangle$ and $\mathbf{P}_1 = \langle x_1, y_1, z_1 \rangle$ lie in the plane.
  - Then $\mathbf{P}_1 \cdot \mathbf{n} = a\,x_1 + b\,y_1 + c\,z_1 = d$ since $\mathbf{P}_1$ lies in the plane, and similarly $\mathbf{P}_2 \cdot \mathbf{n} = d$.
  - But then we have $\mathbf{v} \cdot \mathbf{n} = (\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{n} = \mathbf{P}_2 \cdot \mathbf{n} - \mathbf{P}_1 \cdot \mathbf{n} = d - d = 0$, so that $\mathbf{v}$ is orthogonal to $\mathbf{n}$ as claimed.
  - For the converse statement, clearly if $\mathbf{n} = \langle a, b, c \rangle$ then the equation of the plane must be $a\,x + b\,y + c\,z = \square$ for some value of $\square$, by the previous argument.
  - But if we are given a point that lies in the plane, we can plug in to see that $\square = ax_0 + by_0 + cz_0$, and so we have uniquely determined the equation of the plane, and hence the plane. We can rewrite the equation as $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, so this is the desired plane.
  - <u>Remark</u>: This proposition says that a plane is specified by its normal vector along with a point that the plane passes through.

- <u>Example</u>: The plane with normal vector $\langle 2, 3, -1 \rangle$ passing through $(2, 2, 7)$ has equation $\boxed{2(x - 2) + 3(y - 2) - (z - 7) = 0}$.

- Using these two results we can solve a wide variety of problems involving lines and planes.

  - The fundamental ideas to remember when working with lines and planes are: specifying a line requires knowing its direction vector and a point it passes through, and specifying a plane requires knowing its normal vector and a point it passes through.

- <u>Example</u>: Find an equation for the plane containing the vectors $\mathbf{v}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{v}_2 = \langle 1, 2, 3 \rangle$ and passing through the point $P = (1, -1, 1)$.

  - The normal vector to the plane is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, so we can find it by taking their cross product.
  - We get $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{k} = \langle 1, -2, 1 \rangle$.
    * Sanity check: we have $\mathbf{n} \cdot \mathbf{v}_1 = (1)(1) + (-2)(1) + (1)(1) = 0$ and $\mathbf{n} \cdot \mathbf{v}_2 = (1)(1) + (-2)(2) + (1)(3) = 0$.
  - Now we see that the equation for the plane is $1(x - 1) - 2(y + 1) + 1(z - 1) = 0$, which we can rewrite as $\boxed{x - 2y + z = 4}$.

- <u>Example</u>: Find an equation for the plane passing through the three points $P_1 = (3, 0, -1)$, $P_2 = (1, 2, 2)$ and $P_3 = (-2, 1, 4)$.

  - We need the plane's normal vector $\mathbf{n}$. To find it, we need to find two vectors lying in the plane, and then take their cross product to get $\mathbf{n}$. We can get two such vectors as $\mathbf{v}_1 = P_2 - P_1$ and $\mathbf{v}_2 = P_3 - P_1$.

- ○ We have $\mathbf{v}_1 = P_2 - P_1 = \langle -2, 2, 3 \rangle$ and $\mathbf{v}_2 = P_3 - P_1 = \langle -5, 1, 5 \rangle$.
- ○ Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 2 \\ -5 & 1 \end{vmatrix} \mathbf{k} = \langle 7, -5, 8 \rangle$.
  - ∗ Sanity check: $\mathbf{n} \cdot \mathbf{v}_1 = (7)(-5) + (-5)(1) + (8)(5) = 0$ and $\mathbf{n} \cdot \mathbf{v}_2 = (7)(-2) + (-5)(2) + (8)(3) = 0$.
- ○ Thus the equation of the plane is $7(x-3) - 5(y-0) + 8(z+1) = 0$, or $\boxed{7x - 5y + 8z = 13}$.
  - ∗ For an extra error check, we verify that all three points do lie in this plane: we have $7(3) - 5(0) + 8(-1) = 13$, $7(1) - 5(2) + 8(2) = 13$, and $7(-2) - 5(1) + 8(4) = 13$. Hence this is the correct equation.

- • Example: Parametrize the line of intersection of $x - y + 2z = 3$ and $2x + y - z = 0$.

  - ○ The normal vector to each plane will be orthogonal to the line of intersection. Therefore, we can get the direction vector of the line by taking the cross product of the two planes' normal vectors.
  - ○ We have $\mathbf{n}_1 = \langle 1, -1, 2 \rangle$ and $\mathbf{n}_2 = \langle 2, 1, -1 \rangle$. Then the cross product is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -1, 5, 3 \rangle$.
    - ∗ Sanity check: $\mathbf{v} \cdot \mathbf{n}_1 = -1(1) + 5(-1) + 3(2) = 0$ and $\mathbf{v} \cdot \mathbf{n}_2 = -1(2) + 5(1) + 3(-1) = 0$.
  - ○ Now we need to find a point in both planes (since we need a point on the line). We try looking for one with $x = 0$: this requires $-y + 2z = 3$ and $y - z = 0$. Solving yields $y = z = 3$, so $(0, 3, 3)$ is in both planes and thus on the line l.
  - ○ Applying the line parametrization formula gives $\boxed{l : \langle x, y, z \rangle = \langle -t,\, 5t+3,\, 3t+3 \rangle}$.

- • Another common problem is to compute the distance from a point to a plane. It turns out there is a reasonably nice formula for this distance:

- • Proposition (Point-to-Plane Distance): The distance from the point $P = (x_0, y_0, z_0)$ to the plane $ax + by + cz = d$ is equal to $\dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

  - ○ The shortest vector connecting P to the plane will be in the same direction as the normal vector to the plane. In theory, we could use this information to parametrize the line joining P to the plane, find the intersection of the line and plane, and finally compute the distance we seek. However, there is a less messy way using vector projections.
  - ○ Proof: Let Q be any point on the plane $ax + by + cz = d$ and let $\mathbf{n}$ be the normal vector to the plane.
  - ○ Then the vector $\mathbf{v}$ connecting P to the plane is given by the vector projection of $\mathbf{w} = Q - P$ onto the normal $\mathbf{n}$.
  - ○ Explicitly, $\mathbf{v} = \text{Proj}_{\mathbf{n}}(\mathbf{w}) = \left(\dfrac{\mathbf{n} \cdot \mathbf{w}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n}$. Then the length of $\mathbf{v}$ is $||\mathbf{v}|| = \dfrac{|\mathbf{n} \cdot \mathbf{w}|}{||\mathbf{n}||^2} ||\mathbf{n}|| = \dfrac{|\mathbf{n} \cdot \mathbf{w}|}{||\mathbf{n}||}$.
  - ○ From our earlier results, we can take $\mathbf{n} = \langle a, b, c \rangle$: then $\mathbf{n} \cdot \mathbf{w} = \mathbf{n} \cdot \mathbf{P} - \mathbf{n} \cdot \mathbf{Q} = (ax_0 + by_0 + cz_0) - (d)$, and $||\mathbf{n}|| = \sqrt{a^2 + b^2 + c^2}$. (The fact that $\mathbf{n} \cdot \mathbf{Q} = d$ is just a restatement of the fact that $\mathbf{Q}$ lies in the plane $ax + by + cz = d$.)
  - ○ Then $||\mathbf{v}|| = \dfrac{|\mathbf{n} \cdot \mathbf{w}|}{||\mathbf{n}||} = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$, as claimed.

- • Example: Find the distance from the point $P = (1, 2, 4)$ to the plane $x + 2y - 2z = 3$.

  - ○ The formula gives $\dfrac{|1 \cdot 1 + 2 \cdot 2 - 2 \cdot 4 - 3|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \dfrac{|-6|}{\sqrt{9}} = \boxed{2}$.

- • Example: Find the distance from the point $P = (1, 2, 4)$ to the plane $z = 0$.

  - ○ The plane can be written as $0x + 0y + z = 0$.
  - ○ Then the formula gives $\dfrac{|0 \cdot 1 + 0 \cdot 2 + 1 \cdot 4 - 0|}{\sqrt{0^2 + 0^2 + 1^2}} = \dfrac{|4|}{1} = \boxed{4}$.
  - ○ This result agrees with our intuition: the plane $z = 0$ is the horizontal xy-plane, and so the distance of any point to this plane is simply the absolute value of its z-coordinate.

- <u>Example</u>: Find the distance between the planes $2x + y + 2z = 1$ and $2x + y + 2z = 7$.

  ○ These two planes are parallel since they clearly have the same normal vector $\langle 2, 1, 2 \rangle$.

  ○ To compute the distance between them, we can pick any point on one plane and compute its distance to the other plane.

  ○ Since the point $(0, 1, 0)$ lies on $2x + y + 2z = 1$, the formula gives the distance as $\dfrac{|2 \cdot 0 + 1 \cdot 1 + 2 \cdot 0 - 7|}{\sqrt{2^2 + 1^2 + 2^2}} = \dfrac{|-6|}{\sqrt{9}} = \boxed{2}$.

  ○ <u>Remark</u>: By the same argument, the distance between the two planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is equal to $\dfrac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$.

- <u>Example</u>: Find the distance between the lines $l_1 : \langle x, y, z \rangle = \langle 4t, \, 1 - t, \, 2 - t \rangle$ and $l_2 : \langle x, y, z \rangle = \langle 2 + 2s, \, 3, \, 5 - s \rangle$.

  ○ Imagine the two lines l_1 and l_2 as sitting inside two parallel planes. The distance between the lines will then be the same as the distance between the planes, which we can compute by finding the distance of any arbitrary point in one plane to the other plane (using the point-to-plane distance formula).

  ○ In order to find these two parallel planes, we just need to find their (common) normal vector $\mathbf{n}$, which will be orthogonal to the direction vectors of both lines. So to compute $\mathbf{n}$, we can simply take the cross product of the direction vectors.

  ○ We compute $\mathbf{n} = \langle 4, -1, -1 \rangle \times \langle 2, 0, -1 \rangle = \langle 1, 2, 2 \rangle$, meaning that the planes are of the form $x + 2y + 2z = d$ for some value of d.

  ○ To get the plane containing l_1, choose any point on l_1. Picking $t = 0$ gives $(0, 1, 2)$, and so we get $d_1 = 0 + 2 \cdot 1 + 2 \cdot 2 = 6$, so the first plane is $x + 2y + 2z = 6$.

  ○ Setting $s = 0$ in the equation for l_2 gives the point $(2, 3, 5)$, which lies in the second plane.

  ○ Then by the point-to-plane distance formula, the required distance is $\dfrac{|2 + 2 \cdot 3 + 2 \cdot 5 - 6|}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{|12|}{\sqrt{9}} = \boxed{4}$.

1.3 Vector-Valued Functions, Curves and Motion in 3-Space

- In this section, we extend the results of the previous section and study more general functions and curves in 3-space using the language of vectors. We introduce the notion of a vector-valued function and of its graph, which will be a curve or surface in space, and then analyze the geometry of curves in 3-space from a physical perspective.

1.3.1 Vector-Valued Functions and Curves

- Now that we have discussed vectors, we can discuss functions involving vectors.

- <u>Definition</u>: A <u>vector-valued function</u> of one variable $\mathbf{r}(t)$ is a function whose output is a vector, each of whose components is a function of the parameter t.

  ○ <u>Examples</u>: $\mathbf{r}_1(t) = \langle t^2, 2t \rangle$, $\mathbf{r}_2(t) = \langle t, t, t \rangle$, $\mathbf{r}_3(t) = \langle \cos(t^2), \, e^{2t}, \, \tan^{-1}(\sqrt{t^2 + 1}) \rangle$.

  ○ We have already encountered a few vector-valued functions when we studied lines in 3-space: the parametrization of a line is an example of a vector-valued function.

- We can add and scalar-multiply vector-valued functions in the same manner as normal vectors.

  ○ <u>Example</u>: For $\mathbf{r}_1(t) = \langle e^t, \cos(t), t^2 - 1 \rangle$ and $\mathbf{r}_2(t) = \langle t, 0, -t^2 \rangle$ we have $\mathbf{r}_1(t) + \mathbf{r}_2(t) = \langle e^t + t, \cos(t), -1 \rangle$ and $2\mathbf{r}_2(t) = \langle 2t, 0, -2t^2 \rangle$.

  ○ We will primarily be interested in vector functions of the form $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and $\mathbf{r}(t) = \langle x(t), \, y(t), \, z(t) \rangle$, which have a single input parameter t and output a vector with 2 or 3 coordinates. As we vary t, these functions will trace out parametric curves in 2 or 3-dimensional space (respectively).

○ Later, we will also be interested in vector functions of the form $\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t) \rangle$, for two input parameters s and t. These functions, in general, will describe surfaces in 3-dimensional space as we vary s and t.

- If we graph a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ as t varies, we will obtain a curve in the plane.

  ○ <u>Example</u>: The curve given by $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ traces around the unit circle as t varies. It traces counterclockwise, and completes one revolution after a time interval of 2π.

  ○ <u>Remark</u>: Graphs of vector-valued functions in the plane are often studied in single-variable calculus (where they are usually termed "parametric curves").

- If we graph a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ as t varies, we will obtain a curve in 3-space.

  ○ A parametric curve in 3-space is just the set of points $(x(t), y(t), z(t))$ for some functions $x(t)$, $y(t)$, and $z(t)$.

  ○ <u>Reminder</u>: We are thinking of functions giving parametric curves interchangeably with vector functions: instead of getting a point $(x(t), y(t), z(t))$ as output from a function, we can equally well think of getting the vector output $\langle x(t), y(t), z(t) \rangle$.

- <u>Example</u>: The curve given by $\mathbf{r}(t) = \langle t, t, t \rangle$ is a line passing through the origin with direction vector $\langle 1, 1, 1 \rangle$.

  ○ More generally, as we have already seen, for any constants a_1, a_2, a_3 and b_1, b_2, b_3, the curve given by $\mathbf{r}(t) = (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3)$ will be a line passing through the point (b_1, b_2, b_3) and pointing in the direction of the vector $\langle a_1, a_2, a_3 \rangle$.

- <u>Example</u>: The curve given by $\mathbf{r}(t) = (\sin(t), \cos(t), t)$ is a helix wrapping around the cylinder $x^2 + y^2 = 1$. We can see that as t increases, the x and y parts just trace around a unit circle at constant speed, while z increases at constant speed. Here is a plot of the curve winding around the cylinder, for $0 \le t \le 8\pi$:
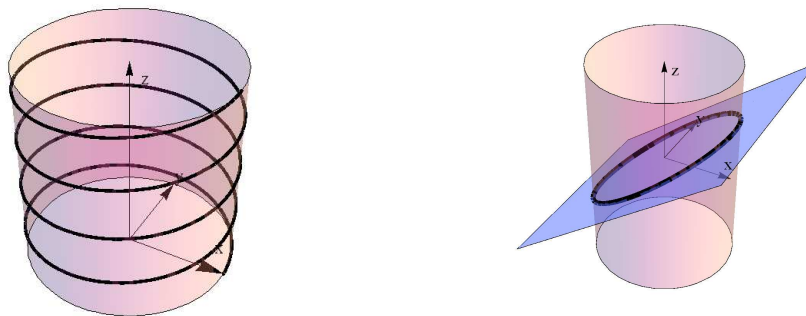
- <u>Example</u>: The curve given by $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos(t) \rangle$ is an ellipse. We can see that this curve is an ellipse by observing that it is the intersection of the plane $z = x$ with the cylinder $x^2 + y^2 = 1$ (above).

- Finding a parametrization of a particular curve with a given description can be difficult, and there is no general recipe. However, in some cases, we can find parametrizations of particular curves:

- <u>Example</u>: Find a parametrization for the curve of intersection between $z = x^2 + y^2$ and $y = x^2$.

  ○ In this case, if we simply try taking $x = t$, then the second equation requires $y = t^2$.

  ○ Once we know $x = t$ and $y = t^2$, the first equation then tells us that $z = t^2 + t^4$, so we get the parametrization $\mathbf{r}(t) = \boxed{\langle t, t^2, t^2 + t^4 \rangle}$. Each point of this form lies on both of the surfaces.

- <u>Example</u>: Find a parametrization for the curve of intersection between $x^2 + z^2 = 4$ and $x + y + z = 3$.

  ○ We can describe all of the points (x, z) satisfying the first equation by taking $x = 2\cos(t)$, $z = 2\sin(t)$.

  ○ Plugging these expressions into the second equation then yields $y = 3 - x - z = 3 - 2\cos(t) - 2\sin(t)$. We obtain the parametrization $\mathbf{r}(t) = \boxed{\langle 2\cos(t), 2\sin(t), 3 - 2\cos(t) - 2\sin(t) \rangle}$.

- The next natural question to ask is: can we take derivatives of vector-valued functions? The answer is yes:

- Definition: The derivative of the vector function $\mathbf{r}(t)$ is given by $\mathbf{r}'(t) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$, provided the limit exists.

  ○ Note the extreme similarity of this definition with the definition of the derivative of a (scalar) function of one variable $f(t)$, which reads $f'(t) = \lim\limits_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$.

  ○ For $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then by applying the definition of the derivative, we see that $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$. Quite sensibly, taking the derivative of a vector function is the same thing as differentiating each component.

  ○ Example: For $\mathbf{r}(t) = \langle e^t, \cos(t), t^2 - 1 \rangle$, we have $\mathbf{r}'(t) = \boxed{\langle e^t, -\sin(t), 2t \rangle}$.

- By the limit part of the definition, we see that the derivative of a vector function tells us in which direction the curve is moving at time t. More specifically, $\mathbf{r}'(t)$ is a vector that is tangent to the graph of $\mathbf{r}(t)$.

  ○ In other words, $\mathbf{r}'(t)$ is the direction vector for the tangent line to the curve parametrized by $\mathbf{r}(t)$.

  ○ Compare to the situation of the derivative of a (scalar) function of one variable $f(t)$: the derivative $f'(t)$ gives the slope of the tangent line.

- Example: Find a parametrization for the tangent line to the curve $\mathbf{r}(t) = \langle t^3, t^4, t^5 \rangle$ at $t = 1$.

  ○ To parametrize the tangent line, we require a point and a direction vector.

  ○ The point of tangency is $\mathbf{r}(1) = \langle 1, 1, 1 \rangle$, while the direction vector is $\mathbf{r}'(1) = \langle 3, 4, 5 \rangle$.

  ○ Therefore, the tangent line has parametrization $\boxed{\langle x, y, z \rangle = \langle 1, 1, 1 \rangle + s \langle 3, 4, 5 \rangle = \langle 1 + 3s, \, 1 + 4s, \, 1 + 5s \rangle}$. (We use the parameter s instead because t was already used to describe the curve.)

- We can also take higher derivatives: for example, $\mathbf{r}''(t) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}'(t+h) - \mathbf{r}'(t)}{h}$, and so forth.

  ○ Example: For $\mathbf{r}(t) = \langle e^t, \cos(t), t^2 - 1 \rangle$, we have $\mathbf{r}'(t) = \langle e^t, -\sin(t), 2t \rangle$, so $\boxed{\mathbf{r}''(t) = \langle e^t, -\cos(t), 2 \rangle}$ and $\boxed{\mathbf{r}'''(t) = \langle e^t, \sin(t), 0 \rangle}$, and so on.

- Derivatives of vector functions satisfy rules strongly reminiscent of the product rule with regard to the dot and cross products: $\dfrac{d}{dt}[\mathbf{r}_1 \cdot \mathbf{r}_2] = \mathbf{r}_1 \cdot (\mathbf{r}_2') + (\mathbf{r}_1') \cdot \mathbf{r}_2$ and $\dfrac{d}{dt}[\mathbf{r}_1 \times \mathbf{r}_2] = \mathbf{r}_1 \times (\mathbf{r}_2') + (\mathbf{r}_1') \times \mathbf{r}_2$.

  ○ These properties can be verified by expanding out the dot and cross products of $[\mathbf{r}_1(t+h) - \mathbf{r}_1(t)]$ with $[\mathbf{r}_2(t+h) - \mathbf{r}_2(t)]$, applying the limit definition of derivative, and simplifying.

1.3.2 Motion in 3-Space: Velocity and Acceleration; Arclength; Unit Tangent, Normal, and Binormal Vectors; Curvature and Torsion

- A particle traveling on a parametric curve $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ in 3-space possesses a number of associated vectors and scalar quantities.

  ○ The velocity of the particle is given by $\mathbf{v}(t) = \mathbf{r}'(t)$, and the speed is the magnitude of the velocity $||\mathbf{v}(t)||$.

  ○ The acceleration of the particle is given by $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

  ○ The arclength of the curve between $t = t_1$ and $t = t_2$ is given by $s = \int_{t_1}^{t_2} ||\mathbf{v}(t)|| \, dt$.

- Example: Find the velocity and acceleration of a particle traveling along the path $\mathbf{r}(t) = \langle 2 + e^t, \, 2 + t\sqrt{2}, \, 3 - e^{-t} \rangle$. Also, find the arclength of the path from $t = 0$ to $t = 1$.

  ○ The velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = \boxed{\langle e^t, \, \sqrt{2}, \, e^{-t} \rangle}$, and the acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \boxed{\langle e^t, \, 0, \, -e^{-t} \rangle}$.

- For the arclength, first we compute $||\mathbf{v}(t)|| = \sqrt{(e^t)^2 + (\sqrt{2})^2 + (e^{-t})^2} = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$.

- Then the arclength is $s = \int_0^1 [e^t + e^{-t}]\, dt = [e^t - e^{-t}]\big|_{t=0}^1 = \boxed{e - e^{-1}}$.

- **Example:** A projectile is fired at time $t = 0$s from initial position $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ m in a vacuum with initial velocity $\mathbf{v}(0) = \langle 2, 3, 45 \rangle$ m/s. Assuming that the only force acting on the projectile is the downward acceleration due to gravity of $\mathbf{a}(t) = \langle 0, 0, -10 \rangle$ m/s², find the position of the projectile at time t, the time it hits the ground, and set up an integral for the total distance it travels along its path.

  - Since the acceleration is $\mathbf{a}(t) = \langle 0, 0, -10 \rangle$ m/s², integrating yields $\mathbf{v}(t) = \langle C_1, C_2, C_3 - 10t \rangle$ m/s for some constants C_1, C_2, C_3.

  - Setting $t = 0$ yields $\mathbf{v}(0) = \langle C_1, C_2, C_3 \rangle$, so since $\mathbf{v}(0)$ is given as $\langle 2, 3, 45 \rangle$ m/s, we see that $\mathbf{v}(t) = \langle 2, 3, 45 - 10t \rangle$ m/s.

  - Integrating again yields $\mathbf{r}(t) = \left\langle C_4 + 2t,\ C_5 + 3t,\ C_6 + 45t - 5t^2 \right\rangle$ m. Since the initial position is $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$ m, we see that $\mathbf{r}(0) = \langle C_4, C_5, C_6 \rangle$, and therefore $\mathbf{r}(t) = \boxed{\left\langle 2t,\ 3t,\ 45t - 5t^2 \right\rangle \text{ m}}$.

  - The height of the projectile is the z-coordinate, so it hits the ground when $45t - 5t^2 = 0$. Factoring gives $5t(9 - t) = 0$, so $t = 0$ or $t = 9$. Since the projectile is fired when $t = 0$s, it hits the ground at time $t = \boxed{9\text{s}}$.

  - The total distance travelled by the particle is the arclength of the path between $t = 0$s and $t = 9$s. Since $||\mathbf{v}(t)|| = \sqrt{2^2 + 3^2 + (45 - 10t)^2}$ m/s, the arclength is given by $\boxed{\int_0^9 \sqrt{2^2 + 3^2 + (45 - 10t)^2}\, dt}$ meters.

  - **Remark:** This integral can be computed using a series of substitutions, although the result is rather messy: $\int_0^9 \sqrt{2^2 + 3^2 + (45 - 10t)^2}\, dt = 9\sqrt{\frac{1019}{2}} + \frac{13}{10}\ln\left[\frac{45 + \sqrt{2038}}{\sqrt{13}}\right] \approx 207.33$.

- When working with curves in 3-space (especially in physics), it is often useful to use normalized tangent vectors, rather than just the velocity and acceleration vectors. There are three standard "normalized tangent vectors": the unit tangent vector $\mathbf{T}(t)$, the unit normal vector $\mathbf{N}(t)$, and the unit binormal vector $\mathbf{B}(t)$.

- **Definition:** If $\mathbf{r}(t)$ represents the position of a particle, the unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{||\mathbf{v}(t)||}$ measures the direction in which the particle is moving.

  - By the definition, the unit tangent vector satisfies $||\mathbf{T}(t)|| = 1$, and $\mathbf{T}(t)$ is in the same direction as $\mathbf{r}'(t)$. Thus, it is a unit vector, in the same direction as the velocity (or "tangent") vector.

- **Definition:** If $\mathbf{r}(t)$ represents the position of a particle, the unit normal vector $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$, orthogonal to the unit tangent vector, measures roughly the direction that forces are pulling the particle.

  - By the definition, the unit normal vector satisfies $||\mathbf{N}(t)|| = 1$, as long as $\mathbf{T}'(t) \neq 0$.

  - To see that $\mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$, since $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$, taking the derivative with respect to t gives $\mathbf{T}(t) \cdot \mathbf{T}'(t) + \mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$. But since the dot product is symmetric, this says $\mathbf{T}$ (the unit tangent vector) and $\mathbf{T}'$ (the normal vector) have dot product 0. In other words, the (unit) normal vector is orthogonal to the tangent vector.

- **Definition:** If $\mathbf{r}(t)$ represents the position of a particle, the unit binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \dfrac{\mathbf{v}(t) \times \mathbf{a}(t)}{||\mathbf{v}(t) \times \mathbf{a}(t)||}$, orthogonal to the tangent and normal vectors, gives the direction perpendicular to the particle's plane of motion.

  - By its definition via the cross product, the binormal vector is orthogonal to both the (unit) normal and (unit) tangent vectors, and since those vectors have length 1, so does $\mathbf{B}$.

  - Thus, we see that the unit tangent, unit normal, and unit binormal vectors are all orthogonal to each other – provided that the normal and binormal vectors are defined (which is not always guaranteed because of the division by $||\mathbf{T}'(t)||$ in the definition of $\mathbf{N}$).

- There are several different planes associated to these normalized tangent vectors.

  ○ The <u>osculating plane</u> is the plane containing $\mathbf{T}(t)$ and $\mathbf{N}(t)$, passing through the point $\mathbf{r}(t)$. It is sometimes called the "tangent plane" to the curve.

  ○ The <u>normal plane</u> is the plane containing $\mathbf{N}(t)$ and $\mathbf{B}(t)$, passing through the point $\mathbf{r}(t)$.

  ○ The <u>rectifying plane</u> is the plane containing $\mathbf{T}(t)$ and $\mathbf{B}(t)$, passing through the point $\mathbf{r}(t)$.

- <u>Example</u>: Find the unit tangent, normal, and binormal vectors for the curve $\mathbf{r}(t) = \langle 3\sin t,\, 5\cos t,\, 4\sin t \rangle$.

  ○ We have $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3\cos t,\, -5\sin t,\, -4\cos t \rangle$, so we get $||\mathbf{v}(t)|| = \sqrt{9\cos^2 t + 25\sin^2 t + 16\cos^2 t} = 5$.

  ○ Therefore, $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{||\mathbf{v}(t)||} = \boxed{\left\langle \dfrac{3}{5}\cos t,\, -\sin t,\, \dfrac{4}{5}\cos t \right\rangle}$.

  ○ Next, we have $\mathbf{T}'(t) = \left\langle -\dfrac{3}{5}\sin t,\, -\cos t,\, -\dfrac{4}{5}\sin t \right\rangle$, so $||\mathbf{T}'(t)|| = \sqrt{\dfrac{9}{25}\sin^2 t + \cos^2 t + \dfrac{16}{25}\sin^2 t} = 1$.

  ○ Therefore, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \boxed{\left\langle -\dfrac{3}{5}\sin t,\, -\cos t,\, -\dfrac{4}{5}\sin t \right\rangle}$.

  ○ Finally, $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{3}{5}\cos t & -\sin t & \dfrac{4}{5}\cos t \\ -\dfrac{3}{5}\sin t & -\cos t & -\dfrac{4}{5}\sin t \end{vmatrix} = \boxed{\left\langle \dfrac{4}{5},\, 0,\, -\dfrac{3}{5} \right\rangle}$.

- Another important quantity, defined in terms of the unit tangent vector, is called the curvature:

- <u>Definition</u>: If $\mathbf{r}(t)$ represents the position of a particle, the <u>curvature</u> $\kappa(t) = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{v}(t)||} = \dfrac{||\mathbf{v}(t) \times \mathbf{a}(t)||}{||\mathbf{v}(t)||^3}$ measures how much the path of the particle is curving.

  ○ Note that the curvature is always nonnegative, since the numerator and denominator are nonnegative.

- <u>Example</u>: Find the curvature of the line $\mathbf{r}(t) = \langle t, 2t, 2t \rangle$.

  ○ We have $\mathbf{v}(t) = \langle 1, 2, 2 \rangle$ so $||\mathbf{v}(t)|| = 3$. Then $\mathbf{T}(t) = \left\langle \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{2}{3} \right\rangle$ and $\mathbf{T}'(t) = \langle 0, 0, 0 \rangle$. Thus, the curvature of this line is always equal to $\boxed{0}$.

  ○ More generally, any line has curvature 0, since for a line $\mathbf{T}(t)$ is always constant, implying that $||\mathbf{T}'(t)|| = 0$. Conversely, from essentially the same observation, the only curves whose curvature is zero everywhere are lines, since a curve with curvature zero must have $\mathbf{T}(t)$ be constant.

- <u>Example</u>: Find the curvature of the circle $\mathbf{r}(t) = \langle r\cos t,\, r\sin t,\, 0 \rangle$.

  ○ We have $\mathbf{v}(t) = \langle -r\sin t,\, r\cos t,\, 0 \rangle$ so $||\mathbf{v}(t)|| = r$ and thus $\mathbf{T}(t) = \langle -\sin t,\, \cos t,\, 0 \rangle$.

  ○ We then have $\mathbf{T}'(t) = \langle -\cos t,\, -\sin t,\, 0 \rangle$, so $||\mathbf{T}'(t)|| = 1$, and therefore $\kappa(t) = \boxed{1/r}$.

  ○ More generally, for any circle of radius r, the curvature is $1/r$.

- From these examples, we can see that a small curvature indicates that the path is close to a line segment, while a large curvature means the path is turning sharply.

  ○ For a general curve, the quantity $1/\kappa(t)$ is called the <u>radius of curvature</u> of the curve: it gives the radius of the "tangent circle" to the curve. (For a circle of radius r, the radius of curvature is also r.)

- The acceleration can be decomposed into a component in the direction of the unit tangent vector $\mathbf{T}$ and a component in the direction of the unit normal vector $\mathbf{N}$. (This decomposition is often used in physics.)

- <u>Definition</u>: The <u>tangential</u> and <u>normal components of acceleration</u> are $a_T = \dfrac{d}{dt}||\mathbf{v}||$ and $a_N = \kappa\,||\mathbf{v}||^2$, and they give the decomposition $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$.

- ○ Note that although a_T can have any sign, a_N is always nonnegative because κ is nonnegative.
- ○ To see that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ for these values of a_T and a_N, observe that $\mathbf{v} = ||\mathbf{v}||\,\mathbf{T}$, and then differentiate both sides using the Product Rule to get $\mathbf{a} = \dfrac{d\,||\mathbf{v}||}{dt}\mathbf{T} + ||\mathbf{v}||\dfrac{d\mathbf{T}}{dt} = \dfrac{d\,||\mathbf{v}||}{dt}\mathbf{T} + ||\mathbf{v}||\,||\mathbf{T}'(t)||\,\mathbf{N}$; then $a_T = \dfrac{d}{dt}\,||\mathbf{v}||$ and $a_N = ||\mathbf{v}||\,||\mathbf{T}'(t)|| = \kappa\,||\mathbf{v}||^2$.
- ○ From $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, one has $\mathbf{v} \times \mathbf{a} = ||\mathbf{v}||\,\mathbf{T} \times (a_T\mathbf{T} + a_N\mathbf{N}) = ||\mathbf{v}||\,a_T\,(\mathbf{T} \times \mathbf{T}) + ||\mathbf{v}||\,a_N\,(\mathbf{T} \times \mathbf{N}) = ||\mathbf{v}||\,a_N\mathbf{B}$. Then one can immediately derive the formulas $\mathbf{B} = \dfrac{\mathbf{v} \times \mathbf{a}}{||\mathbf{v} \times \mathbf{a}||}$ and $||\mathbf{v} \times \mathbf{a}|| = ||\mathbf{v}||\,a_N = \kappa\,||\mathbf{v}||^2$.

- The final quantity associated to a curve's motion is called the torsion:

- Definition: If $\mathbf{r}(t)$ represents a particle's position, the torsion $\tau = -\dfrac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{||\mathbf{v}(t)||} = \dfrac{1}{||\mathbf{v} \times \mathbf{a}||^2}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$

  measures how sharply the path of the particle is twisting out of its plane of motion.

  - ○ The torsion is the least intuitive of all of the quantities we have listed. (Note that it is the only one to involve the third derivatives of the coordinate functions.)
  - ○ Another relation satisfied by the torsion, which may be easier to use, is $-\tau\mathbf{N} = \dfrac{1}{||\mathbf{v}||}\dfrac{d\mathbf{B}}{dt}$.
  - ○ By observing that $\dfrac{ds}{dt} = ||\mathbf{v}||$, we can rewrite the above relation as $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. This is one of the three "Frenet-Serret formulas": the other two say $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ and $\dfrac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}$.
  - ○ The $\dfrac{d\mathbf{T}}{ds}$ formula follows from the definition of κ, while the $\dfrac{d\mathbf{N}}{ds}$ formula can be obtained by writing $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and then taking the derivative with respect to s: one obtains $\dfrac{d\mathbf{N}}{ds} = \dfrac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \dfrac{d\mathbf{T}}{ds} = -\tau(\mathbf{N} \times \mathbf{T}) + \mathbf{B} \times (\kappa\mathbf{N}) = \tau\mathbf{B} - \kappa\mathbf{T}$.

- Example: For the curve $\mathbf{r}(t) = \langle \sin(3t), \cos(3t), 4t \rangle$, find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ, τ, a_T, a_N.

  - ○ We have $\mathbf{v}(t) = \langle 3\cos(3t), -3\sin(3t), 4 \rangle$, so that $||\mathbf{v}|| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 16} = 5$.
  - ○ Thus, we get $\mathbf{T} = \boxed{\left\langle \dfrac{3\cos(3t)}{5}, -\dfrac{3\sin(3t)}{5}, \dfrac{4}{5} \right\rangle}$.
  - ○ Next we compute $\dfrac{d\mathbf{T}}{dt} = \left\langle -\dfrac{9\sin(3t)}{5}, -\dfrac{9\cos(3t)}{5}, 0 \right\rangle$ so that $\left|\left|\dfrac{d\mathbf{T}}{dt}\right|\right| = \dfrac{9}{5}$.
  - ○ Then we see that $\mathbf{N} = \boxed{\langle -\sin(3t), -\cos(3t), 0 \rangle}$.
  - ○ Next, $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{3\cos(3t)}{5} & -\dfrac{3\sin(3t)}{5} & \dfrac{4}{5} \\ -\sin(3t) & -\cos(3t) & 0 \end{vmatrix} = \boxed{\left\langle \dfrac{4\cos(3t)}{5}, -\dfrac{4\sin(3t)}{5}, -\dfrac{3}{5} \right\rangle}$.
  - ○ The curvature is given by $\kappa = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{v}(t)||} = \dfrac{9/5}{5} = \boxed{\dfrac{9}{25}}$.
  - ○ We have $\dfrac{d\mathbf{B}}{dt} = \left\langle -\dfrac{12\sin(3t)}{5}, -\dfrac{12\cos(3t)}{5}, 0 \right\rangle$, so $\tau = -\dfrac{\dfrac{d\mathbf{B}}{dt} \cdot \mathbf{N}}{||v||} = \boxed{\dfrac{12}{25}}$.
  - ○ Finally, we have $a_T = \dfrac{d}{dt}[5] = \boxed{0}$ and $a_N = \kappa\,||\mathbf{v}||^2 = \boxed{9}$.
  - ○ As a sanity check, we compute $\mathbf{a}(t) = \langle -9\sin(3t), -9\cos(3t), 0 \rangle$, and indeed $\mathbf{a}(t) = 0\,\mathbf{T} + 9\,\mathbf{N}$.

Well, you're at the end of my handout. Hope it was helpful.