

1. (a) Take  $\mathbf{r}(t) = \langle 3t, 1+t \rangle$  for  $0 \leq t \leq 1$  with  $ds = \|\mathbf{v}(t)\| dt = \sqrt{10} dt$ . Integral is  $\int_0^1 (3t)^2 \sqrt{10} dt = \boxed{3\sqrt{10}}$ .
- (b) Take  $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$  for  $0 \leq t \leq 2\pi$  with  $dx = -3 \sin t dt$  and  $dy = 3 \cos t dt$ .  
Integral is  $\int_0^{2\pi} (3 \cos t)(-3 \sin t) dt + (3 \sin t)(3 \cos t) dt = \int_0^{2\pi} 0 dt = \boxed{0}$ .
- (c) Have  $ds = \|\mathbf{v}(t)\| = \|\langle -3 \sin 3t, 3 \cos 3t, 4 \rangle\| = 5$  and  $\sqrt{x^2 + y^2} = 1$ . Integral is  $\int_0^\pi 1 \cdot 5 dt = \boxed{5\pi}$ .
- (d) Average value is  $[\int_C y ds] / [\int_C 1 ds]$ . Parametrize by  $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t \rangle$  for  $0 \leq t \leq \pi$  with  $ds = \|\mathbf{v}(t)\| dt = 2 dt$ . Numerator is  $\int_0^\pi 2 \sin t \cdot 2 dt = 8$ , denominator is  $\int_0^\pi 2 dt = 2\pi$ , ratio is  $\boxed{8/(2\pi) = 4/\pi}$ .
- (e) Work is  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz$ . With  $x = t^2, y = t^4, z = t^3$  get  $dx = 2t dt, dy = 4t^3 dt, dz = 3t^2 dt$ ,  
 $P = yz = t^7, Q = xz = t^5, R = x^3 = t^6$ . Integral is  $\int_0^1 t^7 \cdot 2t dt + t^5 \cdot 4t^3 dt + t^6 \cdot 3t^2 dt = \int_0^1 9t^8 dt = \boxed{1 \text{ J}}$ .
- (f) Parametrize by  $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$  for  $0 \leq t \leq \pi$ . With  $x = \cos t, y = \sin t$ , get  $dx = -\sin t dt, dy = \cos t dt$ ,  
 $P = y + x = \sin t + \cos t, Q = y - x = \sin t - \cos t$ .  
Circulation is  $\int_C P dx + Q dy = \int_0^\pi (\sin t + \cos t)(-\sin t dt) + (\sin t - \cos t)(\cos t dt) = \int_0^\pi -1 dt = \boxed{-\pi}$ .  
Flux is  $\int_C -Q dx + P dy = \int_0^\pi (\cos t - \sin t)(-\sin t dt) + (\sin t + \cos t)(\cos t dt) = \int_0^\pi 1 dt = \boxed{\pi}$ .
- (g) With  $x = t^2, y = t^3$  get  $dx = 2t dt, dy = 3t^2 dt, P = xy = t^5, Q = y^2 = t^6$ .  
Circulation is  $\int_C P dx + Q dy = \int_0^1 t^5(2t dt) + t^6(3t^2 dt) = \int_0^1 (2t^6 + 3t^8) dt = \boxed{13/21}$ .  
Flux is  $\int_C -Q dx + P dy = \int_0^1 -t^6(2t dt) + t^5(3t^2 dt) = \int_0^1 t^7 dt = \boxed{1/8}$ .
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2. (a) Normal vector is  $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 1, 2s, 3s^2 \rangle \times \langle 1, 2t, 3t^2 \rangle$ . With  $s = 1$  and  $t = 2$  this is  $\mathbf{n} = \langle 1, 2, 3 \rangle \times \langle 1, 4, 12 \rangle = \langle 12, -9, 2 \rangle$ . Point is  $\mathbf{r}(1, 2) = \langle 3, 5, 9 \rangle$ , so equation is  $\boxed{12(x-3) - 9(y-5) + 2(z-9) = 0}$ .
- (b) If  $(x, y, z) = (1, 2, -3)$  then  $t = -1$  and  $s = -1$ . Then  $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 2s, 2t, 0 \rangle \times \langle 0, 2s, 9t^2 \rangle$ , so  $s = -1$  and  $t = -1$  give  $\mathbf{n} = \langle -2, -2, 0 \rangle \times \langle 0, -2, 9 \rangle = \langle -18, 18, 4 \rangle$ . Equation is  $\boxed{-18(x-1) + 18(y-2) + 4(z+3) = 0}$ .
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3. (a) Parametrize by  $\mathbf{r}(s, t) = \langle s, t, 2s+2t \rangle$  for  $1 \leq s \leq 2, 2 \leq t \leq 4$ . Then  $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle -2, -2, 1 \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\| ds dt = 3 ds dt$ . Surface integral is  $\int_2^4 \int_1^2 (s^2 + t^2) \cdot 3 ds dt$ .
- (b) Parametrize by  $\mathbf{r}(z, \theta) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle$  for  $0 \leq \theta \leq 2\pi, 0 \leq z \leq 4$ . Then  $\frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -2 \cos \theta, 2 \sin \theta, 0 \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| dz d\theta = 2 dz d\theta$ . Surface integral is  $\int_0^{2\pi} \int_0^4 8z dz d\theta$ .
- (c) Parametrize by  $\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 4r \rangle$  for  $0 \leq \theta \leq 2\pi, 0 \leq r \leq 2$ . Then  $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -4r \cos \theta, -4r \sin \theta, r \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| dr d\theta = r\sqrt{17} dr d\theta$ . Surface integral is  $\int_0^{2\pi} \int_0^2 r \cdot r\sqrt{17} dr d\theta$ .
- (d) Parametrize by  $\mathbf{r}(\theta, \varphi) = \langle \cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi \rangle$  for  $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$ . Then  $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} = \langle -\cos \theta \sin^2 \varphi, -\sin \theta \sin^2 \varphi, -\sin \varphi \cos \varphi \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right\| d\varphi d\theta = \sin \varphi d\varphi d\theta$ . Surface integral is then  $\int_0^{2\pi} \int_0^\pi \cos^2 \varphi \sin \varphi d\varphi d\theta$ .
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4. (a) Parametrize by  $\mathbf{r}(s, t) = \langle s, t, 3s+4t+11 \rangle$  for  $0 \leq s \leq 1, 0 \leq t \leq 2$ . Then  $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle -3, -4, 1 \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\| ds dt = \sqrt{26} ds dt$ . Surface area integral is  $\int_0^1 \int_0^2 1 \cdot \sqrt{26} ds dt$ .
- (b) Here  $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle 2s, t, 0 \rangle \times \langle 0, s, 2t \rangle = \langle 2t^2, -4st, 2s^2 \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right\| ds dt = 2\sqrt{s^4 + 4s^2t^2 + t^4} ds dt$ . Surface area integral is  $\int_0^1 \int_0^2 1 \cdot 2\sqrt{s^4 + 4s^2t^2 + t^4} ds dt$ .
- (c) Parametrize by  $\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 3r \rangle$  for  $0 \leq \theta \leq 2\pi, 2 \leq r \leq 3$ . Then  $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -3r \cos \theta, 3r \sin \theta, r \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| dr d\theta = r\sqrt{10} dr d\theta$ . Surface area integral is  $\int_0^{2\pi} \int_2^3 1 \cdot r\sqrt{10} dr d\theta$ .
- (d) Cone is  $\varphi = \pi/6$ , so can parametrize by  $\mathbf{r}(\theta, \varphi) = \langle 3 \cos \theta \sin \varphi, 3 \sin \theta \sin \varphi, 3 \cos \varphi \rangle$  for  $0 \leq \theta \leq 2\pi, \pi/6 \leq \varphi \leq \pi$ . Then  $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} = \langle -9 \cos \theta \sin^2 \varphi, -9 \sin \theta \sin^2 \varphi, -9 \sin \varphi \cos \varphi \rangle$  so  $d\sigma = \left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} \right\| d\varphi d\theta = 9 \sin \varphi d\varphi d\theta$ . Surface area integral is  $\int_0^{2\pi} \int_{\pi/6}^\pi 1 \cdot 9 \sin \varphi d\varphi d\theta$ .
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5. (a) Parametrize as  $\mathbf{r}(s, t) = \langle s, t, 4 - s - t \rangle$  for  $0 \leq s \leq 1$ ,  $0 \leq t \leq 2$ . Then  $\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \langle 1, 1, 1 \rangle$  and  $\mathbf{F} = \langle s^2, 2s^2, 3s^2 \rangle$ . Surface integral is  $\int_0^2 \int_0^1 \langle 1, 1, 1 \rangle \cdot \langle s^2, 2s^2, 3s^2 \rangle ds dt = \int_0^2 \int_0^1 6s^2 ds dt = \boxed{4}$ .
- (b) Parametrize as  $\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r \rangle$  for  $0 \leq r \leq \sqrt{6}$ ,  $0 \leq \theta \leq 2\pi$ . Then  $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r \cos \theta, -r \sin \theta, r \rangle$  and  $\mathbf{F} = \langle -r^2 \cos \theta, -r^2 \sin \theta, r^2 \rangle$ . Surface integral is  $\int_0^{2\pi} \int_0^{\sqrt{6}} \langle -r \cos \theta, -r \sin \theta, r \rangle \cdot \langle -r^2 \cos \theta, -r^2 \sin \theta, r^2 \rangle dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{6}} 2r^3 dr d\theta = \boxed{36\pi}$ .
- (c) Parametrize as  $\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle$  for  $0 \leq \theta \leq 2\pi$ ,  $0 \leq z \leq 3$ . Then  $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle$  and  $\mathbf{F} = \langle 2z \cos \theta, 2z \sin \theta, z^4 \rangle$ . Surface integral is  $\int_0^{2\pi} \int_0^3 \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle \cdot \langle 2z \cos \theta, 2z \sin \theta, z^4 \rangle dz d\theta = \int_0^{2\pi} \int_0^3 4z dz d\theta = \boxed{36\pi}$ .
- (d) Parametrize as  $\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \rangle$  for  $0 \leq \theta \leq 2\pi$ ,  $0 \leq r \leq \sqrt{2}$ . Then  $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle -2r^2 \cos \theta, -2r^2 \sin \theta, r \rangle$  and  $\mathbf{F} = \langle r \sin \theta, -r \cos \theta, r^2 \rangle$ . Surface integral is  $\int_0^{2\pi} \int_0^{\sqrt{2}} \langle -2r^2 \cos \theta, -2r^2 \sin \theta, r \rangle \cdot \langle r \sin \theta, -r \cos \theta, r^2 \rangle dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} r^3 dr d\theta = \boxed{2\pi}$ .
- (e) Parametrize by  $\mathbf{r}(\theta, \varphi) = \langle \cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi \rangle$  for  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \varphi \leq \pi/2$ . Then  $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi} = \langle \cos \theta \sin^2 \varphi, \sin \theta \sin^2 \varphi, \sin \varphi \cos \varphi \rangle$  and  $\mathbf{F} = \langle \cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi \rangle$ . Integral is then  $\int_0^{2\pi} \int_0^{\pi/2} \langle \cos \theta \sin^2 \varphi, \sin \theta \sin^2 \varphi, \sin \varphi \cos \varphi \rangle \cdot \langle \cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi \rangle d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi/2} \sin \varphi d\varphi d\theta = \boxed{2\pi}$ .
- (f) Parametrize by  $\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, 1 - r^2 \rangle$  for  $0 \leq \theta \leq 2\pi$ ,  $0 \leq r \leq 1$ . Then  $\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \langle 2r^2 \cos \theta, 2r^2 \sin \theta, r \rangle$  and  $\mathbf{F} = \langle r^3 \sin^3 \theta, r^3 \cos^3 \theta, 1 \rangle$ . Integral is then  $\int_0^{2\pi} \int_0^1 \langle 2r^2 \cos \theta, 2r^2 \sin \theta, r \rangle \cdot \langle r^3 \sin^3 \theta, r^3 \cos^3 \theta, 1 \rangle dr d\theta = \int_0^{2\pi} \int_0^1 (2r^5 \sin \theta \cos \theta + r) dr d\theta = \boxed{\pi}$ .

6. (a)  $\nabla \cdot \mathbf{F} = \boxed{3x^2 + y + 3y^2 + x}$ ,  $\nabla \times \mathbf{F} = \boxed{\langle 0, 0, y - x \rangle}$ . Since  $\nabla \times \mathbf{F} \neq \mathbf{0}$ , field is **not conservative**.
- (b)  $\nabla \cdot \mathbf{F} = \boxed{2}$ ,  $\nabla \times \mathbf{F} = \boxed{\langle 0, 0, 0 \rangle}$ . Since  $\nabla \times \mathbf{F} = \mathbf{0}$ , field is **conservative**. A potential is  $U = \boxed{xyz + x^2 + 2yz}$ .
- (c)  $\nabla \cdot \mathbf{F} = \boxed{2xyz + 2x^2y}$ ,  $\nabla \times \mathbf{F} = \boxed{\langle 0, x^2y - 4xyz, 2xz^2 - x^2z \rangle}$ . Since  $\nabla \times \mathbf{F} \neq \mathbf{0}$ , field is **not conservative**.
- (d)  $\nabla \cdot \mathbf{F} = \boxed{10yz + xe^z}$ ,  $\nabla \times \mathbf{F} = \boxed{\langle 0, 0, 0 \rangle}$ . Since  $\nabla \times \mathbf{F} = \mathbf{0}$ , field is **conservative**. A potential function is  $U = \boxed{x^2yz + xe^z + 3y + 2z}$ .

7. Integral is work done by  $\mathbf{F} = \langle yz, xz, xy \rangle$  along  $C$  from  $\mathbf{r}(0) = \langle 0, 0, 0 \rangle$  to  $\mathbf{r}(1) = \langle e, \pi/4, \ln(2) \rangle$ .  $\mathbf{F}$  is conservative with potential  $U = xyz$ , so work is simply  $U(e, \pi/4, \ln(2)) - U(0, 0, 0) = \boxed{e \cdot (\pi/4) \cdot \ln(2)}$ .

8. By Green's theorem,  $\oint_C P dx + Q dy = \iint_R (Q_x - P_y) dy dx = \int_0^1 \int_0^1 (2y^2 - y - 1) dy dx = \boxed{-5/6}$ .

9. By Green, circulation  $\oint_C P dx + Q dy = \iint_R (Q_x - P_y) dA$  and flux  $\oint_C -Q dx + P dy = \iint_R (P_x + Q_y) dA$ .

- (a) Region is  $0 \leq x \leq 2$ ,  $0 \leq y \leq 3$ . Circ is  $\int_0^2 \int_0^3 (-2xy) dy dx = \boxed{-18}$ , flux is  $\int_0^2 \int_0^3 4y^2 dy dx = \boxed{72}$ .
- (b) Region is  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2 - 2x$ . Circ is  $\int_0^1 \int_0^{2-2x} (-6x) dy dx = \boxed{-2}$ , flux is  $\int_0^1 \int_0^{2-2x} 6y dy dx = \boxed{4}$ .
- (c) Region is  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq 2\pi$ . Circ is  $\int_0^{2\pi} \int_0^1 1 \cdot r dr d\theta = \boxed{\pi}$ , flux is  $\int_0^{2\pi} \int_0^1 7 \cdot r dr d\theta = \boxed{7\pi}$ .
- (d) Region is  $0 \leq r \leq 4$ ,  $0 \leq \theta \leq \pi/2$ . Circ is  $\int_0^{\pi/2} \int_0^4 3r^2 \cdot r dr d\theta = \boxed{96\pi}$ , flux is  $\int_0^{\pi/2} \int_0^4 3r^2 \cdot r dr d\theta = \boxed{96\pi}$ .

10. Note that  $C$  is the counterclockwise boundary of the polar region  $0 \leq r \leq 2$  and  $\pi/4 \leq \theta \leq \pi$ .

(a) By Green, flux of  $\mathbf{F} = \langle P, Q \rangle$  is  $\iint_R (P_x + Q_y) dA = \int_{\pi/4}^{\pi} \int_0^2 0 \cdot r dr d\theta = \boxed{0}$ .

(b) By Green, circulation of  $\mathbf{F} = \langle P, Q \rangle$  is  $\iint_R (Q_x - P_y) dA = \int_{\pi/4}^{\pi} \int_0^2 3r^2 \cdot r dr d\theta = \boxed{9\pi}$ .

(c) We can use the tangential form of Green's theorem for the work integral, since it is the same as the circulation integral. So the work is  $\iint_R (Q_x - P_y) dA = \int_{\pi/4}^{\pi} \int_0^2 4 \cdot r dr d\theta = \boxed{6\pi}$ .

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11. (a) We compute  $\text{curl}(\mathbf{F}) = \langle 0, 0, -5 \rangle$ . Since this is nonzero,  $\mathbf{F}$  is not conservative.

(b) We can use Green's theorem since this path is the counterclockwise boundary of the polar region  $0 \leq r \leq 4$  and  $\pi/2 \leq \theta \leq \pi$ . By Green, the work is  $\iint_R (Q_x - P_y) dA = \int_{\pi/2}^{\pi} \int_0^4 -5 \cdot r dr d\theta = \boxed{-20\pi \text{ J}}$ .

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12. This path is not closed because it is missing the segment from  $(4, 0)$  to  $(0, 0)$ : that segment is parametrized by  $\mathbf{r}(t) = \langle 4 - 4t, 0 \rangle$  for  $0 \leq t \leq 1$  so the line integral on that segment is  $\int_C (1 - y) dx + (e^{y^2} + 2x) dy = \int_0^1 1 \cdot (-4 dt) + (9 - 8t) \cdot (0 dt) = -2$ . If we add that segment back in, we could then use Green's theorem to evaluate the integral along the full path as  $\iint_R (Q_x - P_y) dA = \int_0^4 \int_0^3 (3) dy dx = 36$ . Therefore, the integral on the three requested pieces is equal to the difference  $36 - (-2) = \boxed{38}$ .

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