Math 2321 (Multivariable Calculus)

Lecture #38 of 37 $\sim$ April ??th, 2021

Final Exam Review #2

Final Exam Topics

The topics for the final exam are as follows:

- Vectors, dot + cross products
- Lines and planes in 3-space
- Curves and motion in 3-space
- Partial derivatives
- Directional derivatives and gradients of functions
- Tangent lines and planes
- The multivariable chain rule
- Linearization
- Minima/maxima/saddle pts
- Optimization on a region
- Lagrange multipliers
- Double integrals in rectangular and polar
- Changing order of integration
- Triple integrals in rectangular, cylindrical, and spherical
- Areas, volumes, mass, center of mass
- Line and surface integrals
- Work, circulation, and flux
- Conservative fields, potential functions, fundamental theorem of line integrals
- Divergence and curl
- Green's theorem
- Stokes's theorem
- The divergence theorem

Exam Information

The exam format is similar to the midterms.

- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- The exam is approximately twice the length of a midterm, and all problems are free-response.
- Unless you have made prior arrangements, the exam is from 10:30am–12:30pm on Thursday, April 29th.
- The official exam time limit is 120 minutes, plus 20 minutes of turnaround time (not to be used for working).
- **LATE SUBMISSIONS WILL BE HEAVILY PENALIZED**. Do not submit the exam late.

Collaboration of any kind is not allowed. If you have any questions during the exam, email me immediately.

Review Problems, I

(Fa18-#1) Find an equation for the tangent plane to the graph of $f(x, y) = e^{2(y-1)}\sqrt{x}$ at the point $(x, y) = (4, 1)$.

Review Problems, I

(Fa18-#1) Find an equation for the tangent plane to the graph of $f(x, y) = e^{2(y-1)}\sqrt{x}$ at the point $(x, y) = (4, 1)$.

- The normal vector to the tangent plane of an implicit surface $g(x, y, z) = c$ at a point P is the gradient $\nabla g(P)$.
- Here, we have the implicit surface $z = e^{2(y-1)}\sqrt{x}$, which we can write as $e^{2(y-1)}\sqrt{x} - z = 0$. If $(x, y) = (4, 1)$ then $z = e^{2(1-1)}\sqrt{4} = 2$, so $P = (4, 1, 2)$.
- Thus, with $g(x, y, z) = e^{2(y-1)}\sqrt{x} - z$, we have $\nabla g = \langle \frac{1}{2}x^{-1/2}e^{2(y-1)}, 2e^{2(y-1)}\sqrt{x}, -1\rangle$, and so $\nabla g(P) = \langle 1/4, 4, -1\rangle$.
- Then the tangent plane is $\boxed{1/4(x-4) + 4(y-1) - 1(z-2) = 0}$.

Review Problems, II

(Fa18-#2) An electric dipole generates an electrostatic potential given by $V(x, y) = \dfrac{y}{x^2 + y^2}$ volts, with x, y in meters.

1. What is the gradient of the potential V at $(x, y) = (1, 2)$?
2. The level curves of V are called equipotential curves. At the point $(1, 2)$, find the direction of a vector which is tangent to the equipotential curve passing through that point. Give your answer as a unit vector with positive **i**-component.

Review Problems, II

(Fa18-#2) An electric dipole generates an electrostatic potential given by $V(x, y) = \dfrac{y}{x^2 + y^2}$ volts, with x, y in meters.

1. What is the gradient of the potential V at $(x, y) = (1, 2)$?
2. The level curves of V are called equipotential curves. At the point $(1, 2)$, find the direction of a vector which is tangent to the equipotential curve passing through that point. Give your answer as a unit vector with positive **i**-component.

- $\nabla V = \left\langle \frac{2xy}{(x^2+y^2)^2}, \frac{x^2-y^2}{(x^2+y^2)^2} \right\rangle$, so $\nabla V(1, 2) = \boxed{\langle \frac{4}{25}, -\frac{3}{25} \rangle}$ V/m.
- The gradient is normal to the tangent curve, so we want a vector $\langle a, b \rangle$ that is perpendicular to the gradient.
- This requires $\langle a, b \rangle \cdot \langle 4/25, -3/25 \rangle = 0$ so that $(4a - 3b)/25 = 0$, so we can take $\langle a, b \rangle = \langle 3, 4 \rangle$. This vector has length 5, so the desired unit vector is $\boxed{\langle 3/5, 4/5 \rangle}$.

Review Problems, III

(Fa18-#3) Find the critical points of $f(x, y) = x^3 - 3xy + y^3$, and classify each critical point as a point where f has a local maximum value, a local minimum value, or a saddle point.

Review Problems, III

(Fa18-#3) Find the critical points of $f(x, y) = x^3 - 3xy + y^3$, and classify each critical point as a point where f has a local maximum value, a local minimum value, or a saddle point.

- First we find the critical points and then we classify them.
- We have $f_x = 3x^2 - 3y$ and $f_y = -3x + 3y^2$.
- Solving $f_x = 0$ yields $y = x^2$ and so $f_y = 0$ becomes $-3x + 3x^4 = 0$, which has solutions $x = 0, 1$.
- Since $y = x^2$, there are two critical points: $\boxed{(0, 0) \text{ and } (1, 1)}$.
- To classify them with the second derivatives test we compute $D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - 3^2$.
- At $(0, 0)$ we have $D = -9$ so this is a $\boxed{\text{saddle point}}$.
- At $(1, 1)$, $D = 27$ and $f_{xx} = 6$ so this is a $\boxed{\text{local minimum}}$.

Review Problems, IV

(Fa18-#4) A box-shaped building with a rectangular base is to have a volume of 8000ft^3. Annual heating and cooling costs will amount to $\$2/\text{ft}^2$ for its roof, front wall, and back wall, and $\$4/\text{ft}^2$ for the two remaining walls. Note that the floor is excluded. What dimensions of the building would minimize these annual costs?

Review Problems, IV

(Fa18-#4) A box-shaped building with a rectangular base is to have a volume of 8000ft^3. Annual heating and cooling costs will amount to $\$2/\text{ft}^2$ for its roof, front wall, and back wall, and $\$4/\text{ft}^2$ for the two remaining walls. Note that the floor is excluded. What dimensions of the building would minimize these annual costs?

- Suppose the dimensions are x feet, y feet, and z feet. Then $xyz = 8000$ and we want to minimize $2xy + 2(2xz) + 2(4yz) = 2xy + 4xz + 8yz$.
- We use Lagrange multipliers: this yields the system $2y + 4z = \lambda yz$, $2x + 8z = \lambda xz$, $4x + 8y = \lambda xy$, $xyz = 8000$.
- After dividing by yz, xz, xy, the first three equations are equivalent to $2/z + 4/y = \lambda$, $2/z + 8/x = \lambda$, $4/y + 8/x = \lambda$.
- Thus $2/z = 4/y = 8/x = \lambda/2$ so $z = 1/\lambda$, $y = 2/\lambda$, $x = 4/\lambda$. The last equation then yields $8/\lambda^3 = 8000$ so $\lambda = 1/10$ and thus $(x, y, z) = \boxed{(40\,\text{ft},\ 20\,\text{ft},\ 10\,\text{ft})}$: this is the minimum.

Review Problems, V

(Fa18-#5) Let R be the filled-in triangle in the first quadrant of the xy-plane bounded by the y-axis, the line $y = x$, and the line $y = 1$. Let T be the solid region in $\mathbb{R}^3$ that lies above R and below the surface $z = 12y^2 - 12x^2 + 24$. Find the volume of T.

Review Problems, V

(Fa18-#5) Let R be the filled-in triangle in the first quadrant of the xy-plane bounded by the y-axis, the line $y = x$, and the line $y = 1$. Let T be the solid region in $\mathbb{R}^3$ that lies above R and below the surface $z = 12y^2 - 12x^2 + 24$. Find the volume of T.

- We can compute the volume as the double integral $\iint_R (12y^2 - 12x^2 + 24)\,dy\,dx$ since the function is always positive above the region.
- A quick sketch of the region shows that with integration order $dx\,dy$ we have $0 \le y \le 1$ and $0 \le x \le y$, so the desired integral is $\int_0^1 \int_0^y (12y^2 - 12x^2 + 24)\,dx\,dy =$ $\int_0^1 (12xy^2 - 4x^3 + 24x)\Big|_{x=0}^{y} dy = \int_0^1 (8y^3 + 24y)\,dy = \boxed{14}$.

Review Problems, VI

(Fa18-#6) Evaluate $\displaystyle\int_0^{3/\sqrt{2}} \int_x^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} y \, dz \, dy \, dx$.

Review Problems, VI

(Fa18-#6) Evaluate $\displaystyle\int_0^{3/\sqrt{2}} \int_x^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} y \, dz \, dy \, dx$.

- This actually isn't so hard to evaluate directly, but it is nicer in cylindrical coordinates.
- In cylindrical, the z-limits are $0 \le z \le 9 - r^2$, and the xy-limits are $0 \le x \le 3/\sqrt{2}$ and $x \le y \le \sqrt{9 - x^2}$, which corresponds to $0 \le r \le 3$ and $\pi/4 \le \theta \le \pi/2$.
- Since $y = r\sin\theta$, in cylindrical the integral is $\displaystyle\int_{\pi/4}^{\pi/2} \int_0^3 \int_0^{9-r^2} r\sin\theta \cdot r \, dz \, dr \, d\theta$ $\displaystyle = \int_{\pi/4}^{\pi/2} \int_0^3 (9r^2 - r^4)\sin\theta \, dr \, d\theta = \int_{\pi/4}^{\pi/2} (162/5)\sin\theta \, d\theta =$ $\boxed{81\sqrt{2}/5}$.

Review Problems, VII

(Fa18-#7) Let S be the portion of the cone given by $\varphi = 5\pi/6$ inside the sphere of radius 5. This surface can be parameterized by $\mathbf{r}(u, v) = \langle \frac{1}{2} u \cos v, \frac{1}{2} u \sin v, -\frac{\sqrt{3}}{2} u \rangle$ for $0 \leq u \leq 5$, $0 \leq v \leq 2\pi$. Find the surface area of S.

Review Problems, VII

(Fa18-#7) Let S be the portion of the cone given by $\varphi = 5\pi/6$ inside the sphere of radius 5. This surface can be parameterized by $\mathbf{r}(u,v) = \langle \frac{1}{2}u\cos v, \frac{1}{2}u\sin v, -\frac{\sqrt{3}}{2}u\rangle$ for $0 \le u \le 5$, $0 \le v \le 2\pi$. Find the surface area of S.

- The surface area is given by $\iint_S 1\, d\sigma$.
- We need $d\sigma = ||(\partial \mathbf{r}/\partial u) \times (\partial \mathbf{r})/\partial v)||\, du\, dv$.
- We have $(\partial \mathbf{r}/\partial u) \times (\partial \mathbf{r})/\partial v) = \langle \frac{1}{2}\cos v, \frac{1}{2}\sin v, -\frac{\sqrt{3}}{2}\rangle \times \langle -\frac{1}{2}u\sin v, \frac{1}{2}u\cos v, 0\rangle = \langle \frac{\sqrt{3}}{4}u\cos v, \frac{\sqrt{3}}{4}u\sin v, \frac{1}{4}u\rangle$.
- Thus $d\sigma = ||\langle \frac{\sqrt{3}}{4}u\cos v, \frac{\sqrt{3}}{4}u\sin v, \frac{1}{4}u\rangle||\, dv\, du = u/2\, du\, dv$.
- The surface area is then $\displaystyle\int_0^{2\pi}\int_0^5 (u/2)\, du\, dv = \boxed{25\pi/2}$.

Review Problems, VIII

(Fa18-#8) Consider $\mathbf{F}(x, y) = \langle 2x^3y^4 + x, 2x^4y^3 + y^2 \rangle$.

1. Show that $\mathbf{F}$ is conservative.
2. Find a potential function f of $\mathbf{F}$.
3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the half-circle of radius 2 centered at the origin going clockwise from $(0, -2)$ to $(0, 2)$.

Review Problems, VIII

(Fa18-#8) Consider $\mathbf{F}(x,y) = \langle 2x^3y^4 + x, 2x^4y^3 + y^2 \rangle$.

1. Show that $\mathbf{F}$ is conservative.
2. Find a potential function f of $\mathbf{F}$.
3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the half-circle of radius 2 centered at the origin going clockwise from $(0,-2)$ to $(0,2)$.

- $\mathbf{F}$ is conservative because it is defined everywhere and $\mathrm{curl}(\mathbf{F}) = \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, 8x^3y^3 - 8x^3y^3 \rangle = \langle 0, 0, 0 \rangle$.
- The potential has $f_x = 2x^3y^4 + x$ and $f_y = 2x^4y^3 + y^2$ so we can take $f = \boxed{\frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{3}y^3}$.
- By the fundamental theorem of line integrals, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(0,2) - f(0,-2) = \boxed{16/3}$.

Review Problems, IX

(Fa18-#9) Consider $\mathbf{F}(x, y) = \langle \sqrt{1 + x^3}, 2xy \rangle$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve that starts at $(0, 0)$, goes along the x-axis to $(2, 0)$, goes vertically to $(2, 1)$, and then goes horizontally to $(0, 1)$.

Review Problems, IX

(Fa18-#9) Consider $\mathbf{F}(x, y) = \langle \sqrt{1 + x^3}, 2xy \rangle$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve that starts at $(0, 0)$, goes along the x-axis to $(2, 0)$, goes vertically to $(2, 1)$, and then goes horizontally to $(0, 1)$.

- The curve is three sides of a rectangle.
- We can then use Green's theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$ on the entire rectangle and then subtract the integral along the missing side.
- By Green, the integral on the entire rectangle is $\int_0^2 \int_0^1 (Q_x - P_y)\,dy\,dx = \int_0^2 \int_0^1 2y\,dy\,dx = 2$.
- The missing side is parametrized by $x = 0$, $y = 1 - t$ for $0 \le t \le 1$. Then $P = 1$, $Q = 0$, $dx = 0\,dt$, $dy = -dt$, so the integral is $\int_0^1 (1)(0\,dt) + 0(-dt) = 0$.
- Hence the overall value is $2 - 0 = \boxed{2}$.

Review Problems, X

(Fa18-#10) Use the Divergence theorem to find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, z^3/3 \rangle$ across the sphere of radius 1 centered at the origin oriented by the outward pointing normal.

Review Problems, X

(Fa18-#10) Use the Divergence theorem to find the flux of the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, z^3/3 \rangle$ across the sphere of radius 1 centered at the origin oriented by the outward pointing normal.

- By the divergence theorem, the flux is $\iiint_D \mathrm{div}(\mathbf{F})\, dV$.
- In spherical, the region is $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, $0 \le \rho \le 1$, and $\mathrm{div}(\mathbf{F}) = 0 + 0 + z^2 = \rho^2 \cos^2 \varphi$.
- Thus, the integral is $\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \cos^2 \varphi \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \cos^2 \varphi \sin \varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} (1/5) \cos^2 \varphi \sin \varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} (2/15)\, d\theta = \boxed{4\pi/15}.$$

Review Problems, XI

(Fa18-#11) Let M be the portion of the graph of $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward. Let $\mathbf{F}(x, y, z) = \langle x^2 + y^2 + y + z^2, 3z, 5 - x^2 - y^2 \rangle$.

1. Let ∂M be the boundary of M. Find a parametrization of ∂M.
2. Use Stokes' theorem to compute $\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

Review Problems, XI

(Fa18-#11) Let M be the portion of the graph of $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward. Let $\mathbf{F}(x, y, z) = \langle x^2 + y^2 + y + z^2, 3z, 5 - x^2 - y^2 \rangle$.

1. Let ∂M be the boundary of M. Find a parametrization of ∂M.
2. Use Stokes' theorem to compute $\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

- The boundary is the circle $x^2 + y^2 = 1$, $z = 0$ with parametrization $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$ for $0 \le t \le 2\pi$.
- By Stokes, the given integral equals $\int_C P\,dx + Q\,dy + R\,dz$.
- Here, $P = x^2 + y^2 + y + z^2 = 1 + \sin t$, $Q = 3z = 0$, $R = 5 - x^2 - y^2 = 4$, and $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = 0\,dt$.
- So the integral is $\int_0^{2\pi} (1 + \sin t)(-\sin t\,dt) + 0(\cos t\,dt) + 4(0\,dt)$ $= \int_0^{2\pi} (-\sin t - \sin^2 t)\,dt = \boxed{-\pi}$.

Review Problems, XII

(Fa19-#1) Find a standard equation of the tangent plane at the point $(8, 2, 1)$ to the level surface of the function $f(x, y, z) = x - y^2 + z^2$.

Review Problems, XII

(Fa19-#1) Find a standard equation of the tangent plane at the point $(8, 2, 1)$ to the level surface of the function $f(x, y, z) = x - y^2 + z^2$.

- The gradient ∇f is the normal vector to the tangent plane.
- We have $\nabla f = \langle 1, -2y, 2z$ so $\nabla f(8, 2, 1) = \langle 1, -4, 2 \rangle$.
- Thus, the tangent plane has equation $\boxed{1(x-8) - 4(y-2) + 2(z-1) = 0}$.

Review Problems, XIII

(Fa19-#2) Suppose that the xy-plane is occupied by a heated metal plate of temperature $T = T(x, y)$ (in Celsius) at the point (x, y) (in m), and $\dfrac{\partial T}{\partial x}(1, 2) = -1\,^\circ\mathrm{C/m}$ and $\dfrac{\partial T}{\partial y}(1, 2) = 2\,^\circ\mathrm{C/m}$. A path parametrized by $\mathbf{r}(t) = \langle t, \frac{4}{1+t^2} \rangle$ is traced on the plate, with t in seconds. What is the instantaneous rate of change of the temperature along the path at the point $(1, 2)$?

Review Problems, XIII

(Fa19-#2) Suppose that the xy-plane is occupied by a heated metal plate of temperature $T = T(x, y)$ (in Celsius) at the point (x, y) (in m), and $\frac{\partial T}{\partial x}(1, 2) = -1\,^\circ\mathrm{C/m}$ and $\frac{\partial T}{\partial y}(1, 2) = 2\,^\circ\mathrm{C/m}$. A path parametrized by $\mathbf{r}(t) = \langle t, \frac{4}{1+t^2} \rangle$ is traced on the plate, with t in seconds. What is the instantaneous rate of change of the temperature along the path at the point $(1, 2)$?

- This is a chain rule problem: $T'(t) = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$.
- Note $dx/dt = 1$ and $dy/dt = -8t/(1+t^2)^2$.
- Since the point $(1, 2)$ corresponds to $t = 1$, we have $T'(1) = (-1\,^\circ\mathrm{C/m})(1\,\mathrm{m/s}) + (2\,^\circ\mathrm{C/m})(-2\,\mathrm{m/s}) = \boxed{-5^\circ\mathrm{C/s}}$.

Review Problems, XIV

(Fa19-#3) A bug is crawling on the surface of a hot plate on which the temperature at (x, y) is $T(x, y) = 3xe^y + 2y \ln x + y$.

1. If the bug is at $(1, 0)$, in what direction should it move to cool off the fastest? What is the rate at which temperature drops in this direction?
2. If the bug is at $(1, 3)$, what is the rate of change of the temperature with respect to distance if the bug is moving southeast, in the direction of $\langle 1, -1 \rangle$?

Review Problems, XIV

(Fa19-#3) A bug is crawling on the surface of a hot plate on which the temperature at (x, y) is $T(x, y) = 3xe^y + 2y \ln x + y$.

1. If the bug is at $(1, 0)$, in what direction should it move to cool off the fastest? What is the rate at which temperature drops in this direction?
2. If the bug is at $(1, 3)$, what is the rate of change of the temperature with respect to distance if the bug is moving southeast, in the direction of $\langle 1, -1 \rangle$?

- Note $\nabla T = \langle 3e^y + 2y/x, 3xe^y + 2 \ln x + 1 \rangle$.
- The direction of fastest decrease is $-\nabla T(1, 0) = \boxed{-\langle 3, 4 \rangle}$, and the corresponding rate of decrease is $||\nabla T(1, 0)|| = \boxed{5}$.
- The unit vector in the southeast direction is $\mathbf{v} = \langle 1, -1 \rangle / ||\langle 1, -1 \rangle|| = \langle 1, -1 \rangle / \sqrt{2}$, and the rate of change is $\nabla T(1, 3) \cdot \mathbf{v} = \langle 3e^3 + 6, 3e^3 + 1 \rangle \cdot \langle 1, -1 \rangle / \sqrt{2} = \boxed{5/\sqrt{2}}$.

Review Problems, XV

(Fa19-#4) Find all four critical points of $f(x, y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2$ and classify each as a local maximum, a local minimum, or a saddle point.

Review Problems, XV

(Fa19-#4) Find all four critical points of $f(x, y) = 2x^3 + 9xy^2 + 15x^2 + 27y^2$ and classify each as a local maximum, a local minimum, or a saddle point.

- First we find the critical points and then we classify them.
- We have $f_x = 6x^2 + 9y^2 + 30x$ and $f_y = 18xy + 54y$. Factoring $f_y = 0$ gives $18y(x + 3) = 0$ so $y = 0$ or $x = -3$.
- If $y = 0$ the first equation gives $6x^2 + 30x = 0$ so $x = 0, -5$. If $x = -3$ the first equation gives $9y^2 - 36 = 0$ so $y = -2, 2$.
- So the critical points are $\boxed{(0, 0), (-5, 0), (-3, -2), (-3, 2)}$.
- To classify, $D = f_{xx}f_{yy} - f_{xy}^2 = (12x + 30)(18x + 54) - (18y)^2$.
- At $(0, 0)$, $D > 0$, $f_{xx} > 0$ so this is a $\boxed{\text{local minimum}}$.
- At $(-5, 0)$, $D > 0$, $f_{xx} < 0$ so this is a $\boxed{\text{local maximum}}$.
- At $(-3, \pm 2)$, $D < 0$, so these are $\boxed{\text{saddle points}}$.

Review Problems, XVI

(Fa19-#5) Find the global maximum of the function $f(x, y) = 3x^2 + xy + 2y^2$ over the filled-in triangle with vertices $(-1, 0)$, $(1, 0)$, and $(0, 2)$.

Review Problems, XVI

(Fa19-#5) Find the global maximum of the function $f(x, y) = 3x^2 + xy + 2y^2$ over the filled-in triangle with vertices $(-1, 0)$, $(1, 0)$, and $(0, 2)$.

- As $f_x = 6x + y$, $f_y = x + 4y$ there is one critical point $(0, 0)$.
- Segment $(-1, 0)$ to $(1, 0)$: param $x = -1 + 2t$, $y = 0$ for $0 \le t \le 1$, then $f = 12t^2 - 12t + 3$ with $f' = 24t - 12$, zero at $t = 1/2$ yielding $(x, y) = (0, 0)$, also endpoints $(-1, 0)$, $(1, 0)$.
- Segment $(1, 0)$ to $(0, 2)$: param $x = 1 - t$, $y = 2t$ for $0 \le t \le 1$, then $f = 9t^2 - 4t + 3$ with $f' = 18t - 4$, zero at $t = 2/9$ yielding $(x, y) = (7/9, 4/9)$, also endpoints $(1, 0)$, $(0, 2)$.
- Segment $(-1, 0)$ to $(0, 2)$: param $x = -1 + t$, $y = 2t$ for $0 \le t \le 1$, then $f = 13t^2 - 8t + 3$ with $f' = 26t - 8$, zero at $t = 4/13$ yielding $(x, y) = (-9/13, 8/13)$, also endpoints $(-1, 0)$, $(0, 2)$.
- We have $f(0, 0) = 0$, $f(-1, 0) = 3$, $f(1, 0) = 3$, $f(7/9, 4/9) = 23/9$, $f(0, 2) = 8$, $f(-9/13, 8/13) = 23/13$. Min is $\boxed{0 \text{ at } (0, 0)}$, max is $\boxed{8 \text{ at } (0, 2)}$.

Review Problems, XVII

(Fa19-#6) Find the volume of the solid below the surface $z = 4 - x^2 - y^2$ and above the xy-plane.

Review Problems, XVII

(Fa19-#6) Find the volume of the solid below the surface $z = 4 - x^2 - y^2$ and above the xy-plane.

- This volume is given by the double integral $\iint_R (4 - x^2 - y^2)\, dA$ over the region R where $4 - x^2 - y^2 \geq 0$.
- We can evaluate this integral by converting to polar coordinates. The region R in polar is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, with function $4 - x^2 - y^2 = 4 - r^2$ and differential $r\, dr\, d\theta$.
- Thus, the integral is $\displaystyle\int_0^{2\pi} \int_0^2 (4 - r^2) \cdot r\, dr\, d\theta$
$$= \int_0^{2\pi} \int_0^2 (4r - r^3)\, dr\, d\theta = \int_0^{2\pi} 4\, d\theta = \boxed{8\pi}.$$
- Alternatively (and essentially equivalently) we could use a triple integral in cylindrical coordinates to find the volume: it is $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 1 \cdot r\, dz\, dr\, d\theta$.

Review Problems, XVIII

(Fa19-#7) Find the mass of the unit ball (i.e. a sphere of radius 1 meter) centered at the origin, with density $\delta(x, y, z) = z^2$ kg/m^3.

Review Problems, XVIII

(Fa19-#7) Find the mass of the unit ball (i.e. a sphere of radius 1 meter) centered at the origin, with density $\delta(x, y, z) = z^2$ kg/m^3.

- The mass is given by $\iiint_D \delta(x, y, z)\, dV$.
- In spherical coordinates, the region is $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, $0 \le \rho \le 1$, the function is $z^2 = \rho^2 \cos^2 \varphi$, and the differential is $\rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$.
- Thus the mass is $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \cos^2 \varphi \cdot \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \cos^2 \varphi \sin \varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} (1/5) \cos^2 \varphi \sin \varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} (2/15)\, d\theta = \boxed{4\pi/15}.$$

Review Problems, XIX

(Fa19-#8) Consider $\mathbf{F}(x, y) = \langle 6x^2 + 2x \sin y, x^2 \cos y + 4y^3 \rangle$.

1. Find the curl of $\mathbf{F}$.
2. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C parametrized by $\mathbf{r}(t) = \langle 2t \cos t, 2t \sin t \rangle$ for $0 \leq t \leq 2\pi$.

Review Problems, XIX

(Fa19-#8) Consider $\mathbf{F}(x,y) = \langle 6x^2 + 2x\sin y, x^2\cos y + 4y^3\rangle$.

1. Find the curl of $\mathbf{F}$.
2. Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C parametrized by $\mathbf{r}(t) = \langle 2t\cos t, 2t\sin t\rangle$ for $0 \le t \le 2\pi$.

- The curl is $\mathrm{curl}(\mathbf{F}) = \langle 0, 0, Q_x - P_y\rangle = \langle 0, 0, 2x\cos y - 2x\cos y\rangle = \langle 0,0,0\rangle$.
- Since the curl is zero, $\mathbf{F}$ is conservative. We can compute the line integral by finding a potential function U with $\nabla U = \mathbf{F}$.
- We need $U_x = 6x^2 + 2x\sin y$ and $U_y = x^2\cos y + 4y^3$, so we can take $U = 2x^3 + x^2\sin y + y^4$.
- Then by the fundamental theorem of line integrals, we have $\int_C \mathbf{F}\cdot d\mathbf{r} = U(\mathbf{r}(2\pi)) - U(\mathbf{r}(0)) = U(4\pi, 0) - U(0,0) = \boxed{128\pi^3}$.

Review Problems, XX

(Fa19-#9) Let C be the circle of radius 3 meters, centered at the origin and oriented counterclockwise. Consider the force field $\mathbf{F}(x, y) = \langle 11x + 7y, 2x + e^{-y^4} \rangle$ newtons, where x and y are in meters. Calculate the work done by $\mathbf{F}$ on an object that moves around the curve C.

Review Problems, XX

(Fa19-#9) Let C be the circle of radius 3 meters, centered at the origin and oriented counterclockwise. Consider the force field $\mathbf{F}(x, y) = \langle 11x + 7y, 2x + e^{-y^4} \rangle$ newtons, where x and y are in meters. Calculate the work done by $\mathbf{F}$ on an object that moves around the curve C.

- This is the work integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$. Since C is a closed curve, by Green's theorem this equals the double integral $\iint_R (Q_x - P_y)\,dA$.
- Since R is the interior of the circle C, we can evaluate the double integral in polar coordinates: the region is $0 \le \theta \le 2\pi$, $0 \le r \le 3$, and the function is $Q_x - P_y = 2 - 7 = -5$, with differential $dA = r\,dr\,d\theta$.
- Thus, the integral is
$$\int_0^{2\pi} \int_0^3 -5r\,dr\,d\theta = \int_0^{2\pi} -45/2\,d\theta = \boxed{-45\pi \text{ J}}.$$

Review Problems, XXI

(Fa19-#10) Find the flux of the vector field $\mathbf{F} = \langle 3x + 1, 2xe^z, 3y^2z + z^3 \rangle$ across the outward oriented faces of a cube **without** the front face at $x = 2$ and with vertices at (0,0,0), (2,0,0), (0,2,0) and (0,0,2).

Review Problems, XXI

(Fa19-#10) Find the flux of the vector field $\mathbf{F} = \langle 3x + 1, 2xe^z, 3y^2 z + z^3 \rangle$ across the outward oriented faces of a cube **without** the front face at $x = 2$ and with vertices at (0,0,0), (2,0,0), (0,2,0) and (0,0,2).

- We can use the divergence theorem here. However, note that the surface is not closed, so we must close it and then subtract the flux through the extra plane.
- We close it by including the plane $x = 2$ with $0 \le y, z \le 2$.
- Then the solid is $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$ and also $\mathrm{div}(\mathbf{F}) = 3 + 3y^2 + 3z^2$.
- Thus by the divergence theorem, the flux through the solid is $\iiint_D \mathrm{div}(\mathbf{F})\, dV = \int_0^2 \int_0^2 \int_0^2 (3 + 3y^2 + 3z^2)\, dz\, dy\, dx = 88.$

Review Problems, XXI

(Fa19-#10) Find the flux of the vector field $\mathbf{F} = \langle 3x + 1, 2xe^z, 3y^2z + z^3 \rangle$ across the outward oriented faces of a cube **without** the front face at $x = 2$ and with vertices at (0,0,0), (2,0,0), (0,2,0) and (0,0,2).

- For the piece being subtracted, which is the portion of the plane $x = 2$ where $0 \leq y \leq 2$ and $0 \leq z \leq 2$, we have a parametrization $\mathbf{r}(s, t) = \langle 2, s, t \rangle$ for $0 \leq s \leq 2$, $0 \leq t \leq 2$.
- Then $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 0, 1, 0 \rangle \times \langle 0, 0, 1 \rangle = \langle 1, 0, 0 \rangle$, which has the correct orientation.
- Then $\mathbf{F} \cdot \mathbf{n} = 7$, and so the surface integral is $\int_0^2 \int_0^2 7 \, dt \, ds = 28$.
- Since the flux across all six planes was 88, that means the flux across the remaining five planes is $88 - 28 = \boxed{60}$.

Review Problems, XXII

(Fa19-#11) Let $\mathbf{F} = \langle 3x^2z - 2y, 3x + 3y^2z^2, 5xe^z + y^2 \rangle$.

1. Find $\mathrm{curl}(\mathbf{F})$.
2. Evaluate $\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where M is the part of the sphere $x^2 + y^2 + z^2 = 25$ below $z = -4$, oriented upwards.

Review Problems, XXII

(Fa19-#11) Let $\mathbf{F} = \langle 3x^2z - 2y, 3x + 3y^2z^2, 5xe^z + y^2 \rangle$.

1. Find $\mathrm{curl}(\mathbf{F})$.
2. Evaluate $\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where M is the part of the sphere $x^2 + y^2 + z^2 = 25$ below $z = -4$, oriented upwards.

- We have $\mathrm{curl}(\mathbf{F}) = \boxed{\langle 2y - 6yz^2, 3x^2 - 5e^z, 3 - (-2) \rangle}$.
- Because S is a surface with boundary C, we use Stokes's theorem: $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C P\,dx + Q\,dy + R\,dz$
- We can parametrize C by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, -4 \rangle$ for $0 \le t \le 2\pi$ (it has correct orientation by the right-hand rule).
- Then $dx = -3\sin t\,dt$, $dy = 3\cos t\,dt$, $dz = 0\,dt$, $P = -6\sin t$, $Q = 9\cos t$, $R = 15\cos t + 9\sin^2 t$.
- Thus by Stokes, the integral is $\int_0^{2\pi} (-6\sin t)(-3\sin t) + (9\cos t)(3\cos t) + (15\cos t + 9\sin^2 t)(0\,dt) = \boxed{45\pi}$.

Review Problems, XXIII

(Sp15-#1) Consider the function $f(x, y) = xe^{xy} + y\sin(x)$. Let P be the point $(2, 0)$ and let $\mathbf{v}$ be the vector from the point $(0, 1)$ to the point $(3, 5)$.

1. Find the directional derivative $D_{\mathbf{u}}(P)$ where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{v}$.
2. Find the direction, as a unit vector, in which f increases most rapidly at P.

Review Problems, XXIV

(Sp15-#1) Consider $f(x, y) = xe^{xy} + y\sin(x)$, let $P = (2, 0)$, and let $\mathbf{v}$ be the vector from $(0, 1)$ to $(3, 5)$.

1. Find the directional derivative $D_{\mathbf{u}}(P)$ where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{v}$.
2. Find the unit vector direction in which f increases most rapidly at P.

Review Problems, XXIV

(Sp15-#1) Consider $f(x, y) = xe^{xy} + y\sin(x)$, let $P = (2, 0)$, and let $\mathbf{v}$ be the vector from $(0, 1)$ to $(3, 5)$.

1. Find the directional derivative $D_{\mathbf{u}}(P)$ where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{v}$.
2. Find the unit vector direction in which f increases most rapidly at P.

- The directional derivative is the dot product of $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$ with the gradient ∇f. Note $\mathbf{v} = \langle 3, 4\rangle$ so $\mathbf{u} = \langle 3, 4\rangle/||\langle 3, 4\rangle|| = \langle \frac{3}{5}, \frac{4}{5}\rangle$.
- Also, $\nabla f = \langle f_x, f_y\rangle = \langle e^{xy} + xye^{xy} + y\cos(x), x^2e^{xy} + \sin(x)\rangle$, so $\nabla f(2, 0) = \langle 1, 4 + \sin(2)\rangle$. So the directional derivative is $\langle 3/5, 4/5\rangle \cdot \langle 1, 4 + \sin(2)\rangle = \boxed{3/5 + 4/5(4 + \sin(2))} \approx 4.527$.
- The direction of fastest increase is in the direction of ∇f.
- We have $\nabla f(2, 0) = \langle 1, 4 + \sin(2)\rangle$, with magnitude $||\nabla f(2, 0)|| = \sqrt{1 + (4 + \sin(2))^2}$.
- So the unit vector is $\boxed{\frac{\langle 1,4+\sin(2)\rangle}{\sqrt{1+(4+\sin(2))^2}}} \approx \langle 0.200, 0.980\rangle$.

Review Problems, XXV

(Sp15-#2) Consider the function $f(x, y, z) = xe^y + xz^2$.

1. Find the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ of the function f at the point $P = (2, 0, 1)$.
2. Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at P.
3. Use the linearization to estimate the value of f at $(1.9, 0.1, 1.5)$.

Review Problems, XXV

(Sp15-#2) Consider the function $f(x, y, z) = xe^y + xz^2$.

1. Find the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ of the function f at the point $P = (2, 0, 1)$.
2. Find the linearization $L(x, y, z)$ of $f(x, y, z)$ at P.
3. Use the linearization to estimate the value of f at $(1.9, 0.1, 1.5)$.

- At $P = (2, 0, 1)$ we have $f_x = e^y + z^2 = \boxed{2}$, $f_y = xe^y = \boxed{2}$, and $f_z = 2xz = \boxed{4}$.
- The linearization $L(x, y, z)$
  $= f(P) + f_x(P)(x-2) + f_y(P)(y-0) + f_z(P)(z-1)$
  $= \boxed{4 + 2(x-2) + 2(y-0) + 4(z-1)}$.
- Then $L(1.9, 0.1, 1.5) = 4 - 0.2 + 0.2 + 2 = \boxed{6}$.

Review Problems, XXVI

(Sp15-#4) Consider the function $f(x, y, z) = x^2 + y^2 + \sqrt{xz}$.

1. Find an equation of the tangent plane to the level surface of f at the point $(1, -1, 1)$.
2. Find the coordinates of the point of intersection of the x-axis with the tangent plane from part 1.

Review Problems, XXVI

(Sp15-#4) Consider the function $f(x,y,z) = x^2 + y^2 + \sqrt{xz}$.

1. Find an equation of the tangent plane to the level surface of f at the point $(1,-1,1)$.
2. Find the coordinates of the point of intersection of the x-axis with the tangent plane from part 1.

- The normal vector to the tangent plane is given by $\nabla f(1,-1,1)$. Since $\nabla f = \langle 2x + \frac{1}{2}(xz)^{-1/2}z, 2y, \frac{1}{2}(xz)^{-1/2}x \rangle$ we have $\nabla f(1,-1,1) = \langle 5/2, -2, 1/2 \rangle$.
- Thus, the tangent plane is $\boxed{(5/2)(x-1) - 2(y+1) + (1/2)(z-1) = 0}$.
- The x-axis is parametrized by $\langle t, 0, 0 \rangle$. Plugging this into the plane's equation gives $(5/2)(t-1) - 2 - 1/2 = 0$ so $t = 2$ and the point is $\boxed{(2,0,0)}$.

Review Problems, XXVII

(Sp15-#5) Find the points at which the function $f(x, y) = 2x^2 + y^2 + 2x^2 y$ attains a local minimum value, a local maximum value, or has a saddle point.

Review Problems, XXVII

(Sp15-#5) Find the points at which the function $f(x, y) = 2x^2 + y^2 + 2x^2y$ attains a local minimum value, a local maximum value, or has a saddle point.

- First we find the critical points and then we classify them.
- We have $f_x = 4x + 4xy$ and $f_y = 2y + 2x^2$, so we get the equations $4x + 4xy = 0$ and $2y + 2x^2 = 0$.
- The second equation gives $y = -x^2$, and then the first equation is $4x - 4x^3 = 0$ with solutions $x = -1, 0, 1$. Thus there are three critical points: $\boxed{(-1, -1),\ (0, 0),\ (1, -1)}$.
- To classify them we use the second derivatives test: we have $D = f_{xx}f_{yy} - f_{xy}^2 = (4 + 4y)(2) - (4x)^2$.
- At $(-1, -1)$, $D = -16$ so this is a $\boxed{\text{saddle point}}$.
- At $(0, 0)$, $D = 8$ and $f_{xx} = 4$ so this is a $\boxed{\text{local minimum}}$.
- At $(1, -1)$, $D = -16$ so this is a $\boxed{\text{saddle point}}$.

Review Problems, XXVIII

(Sp15-#6) Find the global minimum and the global maximum values of the function $f(x, y) = -2 + x^2 + 2y^2$ in the closed disk D where $x^2 + y^2 \leq 2$, as well as the coordinates of the points where these extreme values are attained.

Review Problems, XXVIII

(Sp15-#6) Find the global minimum and the global maximum values of the function $f(x, y) = -2 + x^2 + 2y^2$ in the closed disk D where $x^2 + y^2 \leq 2$, as well as the coordinates of the points where these extreme values are attained.

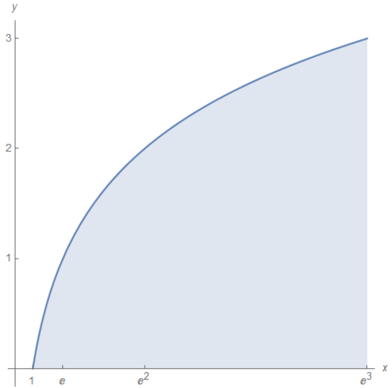
- First we find any critical points, then we analyze the boundary.
- We have $f_x = 2x$ and $f_y = 4y$ so f has one critical point $(0, 0)$.
- The boundary is the circle $x^2 + y^2 = 2$, so we can use Lagrange multipliers with $g(x, y) = x^2 + y^2$.
- Our system is $2x = \lambda \cdot 2x$, $4y = \lambda \cdot 2y$, and $x^2 + y^2 = 2$.
- The first equation gives $x = 0$ or $\lambda = 1$. If $\lambda = 1$ then the second equation gives $y = 0$. So the boundary-critical points occur where $x = 0$ or $y = 0$: these are $(\pm\sqrt{2}, 0)$, $(0, \pm\sqrt{2})$.
- Our full list is $(0, 0)$, $(\pm\sqrt{2}, 0)$, $(0, \pm\sqrt{2})$.
- Since $f(0, 0) = -2$, $f(\pm\sqrt{2}, 0) = 0$, $f(0, \pm\sqrt{2}) = 2$, the $\boxed{\text{min is } -2 \text{ at } (0,0)}$ and the $\boxed{\text{max is 2 at } (0, \pm\sqrt{2})}$.

Review Problems, XXIX

(Sp15-#7) Consider a double integral $\iint_R (x^3 - y^2)\, dA = \int_1^{e^3} \int_0^{\ln x} (x^3 - y^2)\, dy\, dx$. Sketch the region of integration R and then reverse the integration order.

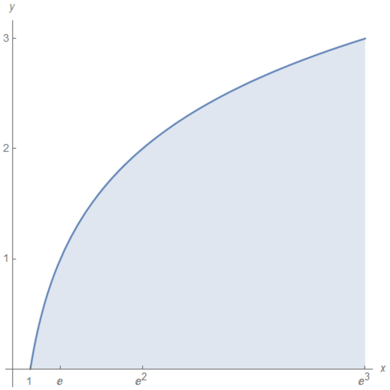
Review Problems, XXIX

(Sp15-#7) Consider a double integral $\iint_R (x^3 - y^2)\,dA = \int_1^{e^3} \int_0^{\ln x} (x^3 - y^2)\,dy\,dx$. Sketch the region of integration R and then reverse the integration order.



Review Problems, XXIX

(Sp15-#7) Consider a double integral $\iint_R (x^3 - y^2)\, dA = \int_1^{e^3} \int_0^{\ln x} (x^3 - y^2)\, dy\, dx$. Sketch the region of integration R and then reverse the integration order.



- The region is $1 \le x \le e^3$, $0 \le y \le \ln x$.
- Using horizontal slices, the slices range from $y = 0$ to $y = 3$, the left curve is $y = \ln x$ aka $x = e^y$, and the right curve is $x = e^3$.
- So the new integral is $\boxed{\int_0^3 \int_{e^y}^{e^3} (x^3 - y^2)\, dy\, dx}$.

Review Problems, XXX

(Sp15-#8) A solid occupies the region S, which is in the 1st octant (where $x, y, z \geq 0$) of the region where $0 \leq z \leq 1$ and $1 \leq x^2 + y^2 \leq 4$. Suppose x, y, z are measured in meters and the solid has density given by $\delta(x, y, z) = \frac{5}{x^2+y^2}\mathrm{kg/m}^3$. Calculate the mass of the solid.

Review Problems, XXX

(Sp15-#8) A solid occupies the region S, which is in the 1st octant (where $x, y, z \geq 0$) of the region where $0 \leq z \leq 1$ and $1 \leq x^2 + y^2 \leq 4$. Suppose x, y, z are measured in meters and the solid has density given by $\delta(x, y, z) = \frac{5}{x^2+y^2}\mathrm{kg}/\mathrm{m}^3$. Calculate the mass of the solid.

- The mass is given by the integral $\iiint_S \delta(x, y, z)\, dV$.
- We can set this triple integral up in cylindrical coordinates: $x, y \geq 0$ corresponds to $0 \leq \theta \leq \pi/2$, and the other conditions give $0 \leq z \leq 1$ and $1 \leq r \leq 2$.
- The density is $(5/r^2)\mathrm{kg}/\mathrm{m}^3$ and the differential is $r\, dz\, dr\, d\theta$.
- Thus, the mass is

$$\int_0^{\pi/2}\int_1^2\int_0^1 \frac{5}{r^2} r\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_1^2\int_0^1 \frac{5}{r}\, dz\, dr\, d\theta$$
$$= \int_0^{\pi/2}\int_1^2 \frac{5}{r}\, dr\, d\theta = \int_0^{\pi/2} 5\ln(2)\, d\theta = \boxed{(5/2)\pi\ln(2)\ \mathrm{kg}}.$$

Review Problems, XXXI

(Sp15-#9) Consider $\mathbf{F}(x, y, z) = \langle ye^z, xe^z + z, xye^z + y + 2z \rangle$.

1. Show that $\mathbf{F}$ is conservative *without* producing a potential function for $\mathbf{F}$.
2. Find a potential function $f(x, y, z)$ for $\mathbf{F}(x, y, z)$.
3. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve in $\mathbb{R}^3$ consisting of straight line segments from $(0, 0, 0)$ to $(1, 1, 0)$, then to $(2, 1, 1)$, and finally to $(3, 2, 1)$.

Review Problems, XXXI

(Sp15-#9) Consider $\mathbf{F}(x, y, z) = \langle ye^z, xe^z + z, xye^z + y + 2z \rangle$.

1. Show that $\mathbf{F}$ is conservative *without* producing a potential function for $\mathbf{F}$.
2. Find a potential function $f(x, y, z)$ for $\mathbf{F}(x, y, z)$.
3. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve in $\mathbb{R}^3$ consisting of straight line segments from $(0, 0, 0)$ to $(1, 1, 0)$, then to $(2, 1, 1)$, and finally to $(3, 2, 1)$.

- $\boxed{\mathbf{F} \text{ is conservative}}$ because it is defined everywhere and its curl is zero: $\mathrm{curl}(\mathbf{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ $= \langle xe^z + 1 - (xe^z + 1, ye^z - ye^z, e^z - e^z \rangle = \langle 0, 0, 0 \rangle$.
- For a potential function we want $f_x = ye^z$, $f_y = xe^z + z$, $f_z = xye^z + y + 2z$. We can take $f = \boxed{xye^z + yz + z^2}$.
- By the fundamental theorem of line integrals, the line integral equals $f(3, 2, 1) - f(0, 0, 0) = \boxed{6e + 3}$.

Review Problems, XXXII

(Sp15-#10) Suppose $\mathbf{F} = \langle 3y + x^2, x^2 + e^{-y^2} \rangle$ represents a force field in newtons, where x and y are in meters.

1. Compute the curl of $\mathbf{F}$.
2. Find the work done by $\mathbf{F}$ on a particle that moves along the curve C given by three sides of a square, starting from $(1, 0)$, to $(1, 1)$, then to $(0, 1)$, and finally to $(0, 0)$.

Review Problems, XXXII

(Sp15-#10) Suppose $\mathbf{F} = \langle 3y + x^2, x^2 + e^{-y^2} \rangle$ represents a force field in newtons, where x and y are in meters.

1. Compute the curl of $\mathbf{F}$.
2. Find the work done by $\mathbf{F}$ on a particle that moves along the curve C given by three sides of a square, starting from $(1, 0)$, to $(1, 1)$, then to $(0, 1)$, and finally to $(0, 0)$.

- We have $\mathrm{curl}(\mathbf{F}) = \langle 0, 0, Q_x - P_y \rangle = \boxed{\langle 0, 0, 2x - 3 \rangle}$.
- For the work, notice that this work equals the work done around the unit square $[0, 1] \times [0, 1]$ minus the work on the segment from $(0, 0)$ to $(1, 0)$.
- By Green's theorem, the work on the whole square equals $\iint_R \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k}\, dA = \int_0^1 \int_0^1 (2x - 3)\, dy\, dx = -2$.
- The segment is parametrized by $x = t$, $y = 0$ for $0 \le t \le 1$ so the work is $\int_C P\, dx + Q\, dy = \int_0^1 t^2\, dt = 1/3$.
- Thus the answer is the difference, $(-2 - 1/3)\,\mathrm{J} = \boxed{-(7/3)\,\mathrm{J}}$.

Review Problems, XXXIII

(Sp15-#12) Consider $\mathbf{F}(x, y, z) = \langle 2y \cos z, e^x \sin z, e^z \rangle$. Let M be the *top hemisphere* of the sphere of radius 3 centered at the origin, oriented upward. Compute the flux integral $\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

Review Problems, XXXIII

(Sp15-#12) Consider $\mathbf{F}(x, y, z) = \langle 2y\cos z, e^x \sin z, e^z\rangle$. Let M be the *top hemisphere* of the sphere of radius 3 centered at the origin, oriented upward. Compute the flux integral $\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

- Setting this up as a surface integral is quite messy because of the exponentials. Instead, we can use Stokes's theorem:
$$\iint_M (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\, dx + Q\, dy + R\, dz$$
where C is the boundary curve of the surface.
- Here, C is the circle $x^2 + y^2 = 9$, $z = 0$ which is parametrized by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 0\rangle$ for $0 \le t \le 2\pi$.
- Then $dx = -3\sin t\, dt$, $dy = 3\cos t\, dt$, $dz = 0\, dt$ and $P = 6\sin t$, $Q = 0$, $R = 1$.
- So, the line integral is
$$\int_0^{2\pi} (6\sin t)(-3\sin t\, dt) + 0 + 0 = \int_0^{2\pi} -18\sin^2 t\, dt = \boxed{-18\pi}.$$

Review Problems, XXXIV

(Sp15-#11) Consider $\mathbf{F}(x, y, z) = \langle \cos z + y^2 e^z, xe^{-z}, z^2 \rangle$. Let M be the part of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ that lies inside the cylinder $x^2 + y^2 = 3$, oriented upward. Compute the flux integral $\iint_M \mathbf{F} \cdot \mathbf{n} \, dS$ of $\mathbf{F}$ through M. Note that M is *not* closed.

Review Problems, XXXIV

(Sp15-#11) Consider $\mathbf{F}(x, y, z) = \langle \cos z + y^2 e^z, xe^{-z}, z^2 \rangle$. Let M be the part of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ that lies inside the cylinder $x^2 + y^2 = 3$, oriented upward. Compute the flux integral $\iint_M \mathbf{F} \cdot \mathbf{n}\, dS$ of $\mathbf{F}$ through M. Note that M is *not* closed.

- If we set up the surface integral directly, it will be very messy because of the exponential terms.
- We can instead try to use the divergence theorem. Here note that $\mathrm{div}(\mathbf{F}) = 0 + 0 + 2z$, which is easy to integrate.
- However, the given surface is not closed, so we will have to add in a surface that will close it, and then subtract off the resulting surface integral.
- A fairly natural surface to use is the disc that lies at the bottom of the spherical slice we have: this is the disc with $x^2 + y^2 \leq 3$ and $z = \sqrt{4 - 3} = 1$.

Review Problems, XXXIV

(Sp15-#11) Consider $\mathbf{F}(x, y, z) = \langle \cos z + y^2 e^z, xe^{-z}, z^2 \rangle$. Let M be the part of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ that lies inside the cylinder $x^2 + y^2 = 3$, oriented upward. Compute the flux integral $\iint_M \mathbf{F} \cdot \mathbf{n}\, dS$ of $\mathbf{F}$ through M. Note that M is *not* closed.

- The integral equals $\iiint_D (\operatorname{div} \mathbf{F})\, dV - \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where D is the solid with $x^2 + y^2 \le 3$ and $1 \le z \le \sqrt{4 - x^2 - y^2}$, and S is the disc $x^2 + y^2 \le 3$, $z = 1$ oriented downward.
- We can set up the triple integral in cylindrical coordinates. The region is $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{3}$, $1 \le z \le \sqrt{4 - r^2}$ with function $(\operatorname{div} \mathbf{F}) = 2z$ and differential $r\, dz\, dr\, d\theta$.
- So the triple integral is $\iiint_D (\operatorname{div} \mathbf{F})\, dV =$

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} 2z \cdot r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} (3r - r^3)\, dr\, d\theta = 9\pi/2.$$

Review Problems, XXXIV

(Sp15-#11) Consider $\mathbf{F}(x, y, z) = \langle \cos z + y^2 e^z, xe^{-z}, z^2 \rangle$. Let M be the part of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ that lies inside the cylinder $x^2 + y^2 = 3$, oriented upward. Compute the flux integral $\iint_M \mathbf{F} \cdot \mathbf{n}\, dS$ of $\mathbf{F}$ through M. Note that M is *not* closed.

- For the surface integral, we parametrize the disc as $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 1 \rangle$ for $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{3}$.
- Then $\mathbf{n} = (\partial \mathbf{r}/\partial r) \times (\partial \mathbf{r}/\partial \theta) = \langle \cos\theta, \sin\theta, 0 \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle = \langle 0, 0, r \rangle$.
- This has the wrong orientation (it needs to be downward) so we need to multiply by -1.
- Then $\mathbf{F} \cdot (-\mathbf{n}) = -r$, so the surface integral is
$$\int_0^{2\pi} \int_0^{\sqrt{3}} -r\, dr\, d\theta = \int_0^{2\pi} (-3/2)\, d\theta = -3\pi.$$
- Thus, the desired surface flux is $(9\pi/2) - (-3\pi) = \boxed{15\pi/2}$.

Final Remarks

I will have office hours after this review until 3:30pm, and also on Wednesday from 1pm-3pm, in case you have any last-minute questions.

This is one of my favorite calculus-level courses[1] to teach, and I hope you enjoyed the course this semester. If you did, please do make sure to fill out the TRACE evaluations.

Happy studying, good luck on the final and on your other exams, have a great summer, and, of course, stay safe!

[1] In summer-2 I am teaching Math 3081 (Probability and Statistics) and in the fall I am teaching Math 2331 (Linear Algebra), in case you have more math courses to take!