Math 2321 (Multivariable Calculus)

Lecture #37 of 37 $\sim$ April 21st, 2021

Final Exam Review #1

Today's review will cover more examples of Stokes's theorem and the divergence theorem.

Final Exam Topics

The topics for the final exam are as follows:

- Vectors, dot + cross products
- Lines and planes in 3-space
- Curves and motion in 3-space
- Partial derivatives
- Directional derivatives and gradients of functions
- Tangent lines and planes
- The multivariable chain rule
- Linearization
- Minima/maxima/saddle pts
- Optimization on a region
- Lagrange multipliers
- Double integrals in rectangular and polar
- Changing order of integration
- Triple integrals in rectangular, cylindrical, and spherical
- Areas, volumes, mass, center of mass
- Line and surface integrals
- Work, circulation, and flux
- Conservative fields, potential functions, fundamental theorem of line integrals
- Divergence and curl
- Green's theorem
- Stokes's theorem
- The divergence theorem

Exam Information

The exam format is similar to the midterms.

- You will write your responses (either on a printout of the exam or on blank paper) and then scan/photograph your responses and upload them into Canvas.
- The exam is approximately twice the length of a midterm, and all problems are free-response.
- Unless you have made prior arrangements, the exam is from 10:30am–12:30pm on Thursday, April 29th.
- The official exam time limit is 120 minutes, plus 20 minutes of turnaround time (not to be used for working).
- Late submissions will be heavily penalized. DO NOT SUBMIT THE EXAM LATE.
- Collaboration of any kind is not allowed.

Review Problems, I

(Rev-#1) Find the flux of $\mathbf{F} = \langle xy^2z^2, \, x^2z^2, \, -xy^2 \rangle$ outward through the surface S made up of the portions of the six planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$, with $0 \leq x, y, z \leq 1$.

Review Problems, I

(Rev-#1) Find the flux of $\mathbf{F} = \langle xy^2z^2, x^2z^2, -xy^2 \rangle$ outward through the surface S made up of the portions of the six planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$, with $0 \leq x, y, z \leq 1$.

- We can use the divergence theorem here, because S is a closed surface enclosing the solid region with $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.
- The theorem says Flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \mathrm{div}(\mathbf{F})\, dV$.
- We have $\mathrm{div}(\mathbf{F}) = P_x + Q_y + R_z = y^2z^2$.
- So, by the divergence theorem, the flux is $\iiint_D \mathrm{div}(\mathbf{F})\, dV = \int_0^1 \int_0^1 \int_0^1 y^2z^2\, dz\, dy\, dx = \boxed{1/9}$.

Review Problems, II

(Rev-#2) Find the flux of $\mathbf{F} = \langle x^3 z, \, y^3 z, \, 0 \rangle$ outward through the boundary of the solid region with $x^2 + y^2 \leq 4$ and $1 \leq z \leq 3$.

Review Problems, II

(Rev-#2) Find the flux of $\mathbf{F} = \langle x^3 z, y^3 z, 0 \rangle$ outward through the boundary of the solid region with $x^2 + y^2 \le 4$ and $1 \le z \le 3$.

- We can use the divergence theorem here, because S is a closed surface enclosing a solid region.
- We can describe the region most easily in cylindrical coordinates: it has $0 \le r \le 2$, $0 \le \theta \le 2\pi$, $1 \le z \le 3$.
- We also have $\mathrm{div}(\mathbf{F}) = 3x^2 z + 3y^2 z = 3r^2 z$ in cylindrical.
- Thus by the divergence theorem, the flux is $\iiint_D \mathrm{div}(\mathbf{F})\, dV = \int_0^{2\pi} \int_0^2 \int_1^3 3r^2 z \cdot r\, dz\, dr\, d\theta = \boxed{96\pi}$.

Review Problems, III

(Rev-#3) Find the flux of $\mathbf{F} = \langle xy^2, yz^2, x^2z \rangle$ through the surface of the unit sphere with outward orientation.

Review Problems, III

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- We can use the divergence theorem here, because S is a closed surface enclosing a solid region.
- We can describe the region most easily in spherical coordinates: it has $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$, $0 \leq \rho \leq 1$
- We also have $\mathrm{div}(\mathbf{F}) = y^2 + z^2 + x^2 = \rho^2$ in spherical.
- Thus by the divergence theorem, the flux is $\iiint_D \mathrm{div}(\mathbf{F})\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \boxed{4\pi/5}$.

Review Problems, IV

(Rev-#4) Find the flux of the curl $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F}(x, y, z) = \langle -2y \cos(z), 2x, xe^y \rangle$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$ with outward orientation.

Review Problems, IV

(Rev-#4) Find the flux of the curl $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F}(x,y,z) = \langle -2y\cos(z), 2x, xe^y \rangle$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$ with outward orientation.

- We can use Stokes's theorem $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C P\,dx + Q\,dy + R\,dz$ here, because S is a surface with boundary C.
- We can parametrize the hemisphere's boundary by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle$ for $0 \le t \le 2\pi$. This has the correct orientation by the right-hand rule.
- Then $dx = -3\sin t\,dt$, $dy = 3\cos t\,dt$, $dz = 0$ and $P = 6\sin t$, $Q = 6\cos t$, $R = 3\cos t \cdot e^{3\sin t}$.
- Thus by Stokes, the integral is $\displaystyle\int_0^{2\pi} (-6\sin t) \cdot (-3\sin t)\,dt + (6\cos t) \cdot 3\cos t\,dt + 3\cos t \cdot e^{3\sin t} \cdot 0\,dt = \int_0^{2\pi} 18\,dt = \boxed{36\pi}$.

Review Problems, V

(Rev-#5) Compute the outward flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ where S is the surface of the "ice cream cone" (the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the sphere $x^2 + y^2 + z^2 = 1$ along with that portion of the sphere that lies above the cone), and $\mathbf{F}(x, y, z) = \left\langle x + 2y^2,\ 5y - 3xz,\ y^2 + 6z \right\rangle$

Review Problems, V

(Rev-#5) Compute the outward flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ where S is the surface of the "ice cream cone" (the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the sphere $x^2 + y^2 + z^2 = 1$ along with that portion of the sphere that lies above the cone), and $\mathbf{F}(x, y, z) = \langle x + 2y^2, 5y - 3xz, y^2 + 6z \rangle$

- We can use the divergence theorem here, because S is a closed surface enclosing a solid region.
- The solid is easiest to describe in spherical: it is $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi/4$, $0 \le \rho \le 1$ in spherical.
- We also have $\mathrm{div}(\mathbf{F}) = 12$.
- Thus by the divergence theorem, the flux is $\iiint_D \mathrm{div}(\mathbf{F})\, dV =$ $\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 12 \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta = \boxed{4\pi(2 - \sqrt{2})}$.

Review Problems, VI

(Rev-#6) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \left\langle x - y^2,\, y - z,\, z^2 + y \right\rangle$ and C is the rectangle with vertices $(0, 0, 4)$, $(2, 0, 0)$, $(2, 1, 0)$, and $(0, 1, 3)$ lying in the plane $x + 2y + z = 4$, oriented counterclockwise from above.

Review Problems, VI

(Rev-#6) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle x - y^2,\, y - z,\, z^2 + y \rangle$ and C is the rectangle with vertices $(0, 0, 4)$, $(2, 0, 0)$, $(2, 1, 0)$, and $(0, 1, 3)$ lying in the plane $x + 2y + z = 4$, oriented counterclockwise from above.

- Use Stokes's theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$ where S is the portion of the plane inside the triangle.
- We can parametrize S as $\mathbf{r}(s, t) = \langle s, t, 4 - s - 2t \rangle$ for $0 \le s \le 2$, $0 \le t \le 1$. Then $\nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle =$ $\langle 1 - (-1),\, 0 - 0,\, 0 - (-2y) \rangle = \langle 2, 0, 2y \rangle$, and so $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 1, 0, -1 \rangle \times \langle 0, 1, -2 \rangle = \langle 1, 2, 1 \rangle$ (correct orientation since the z-coordinate is positive).
- Then $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 2 + 2y = 2 + 2t$ so the surface integral is $\int_0^2 \int_0^1 (2 + 2t)\, dt\, ds = \boxed{6}$.

Review Problems, VII

(Rev-#7) Find the circulation of the vector field $\mathbf{F} = \langle 2xy, x^2, y \rangle$ around the counterclockwise boundary of the portion of the surface $z = x^2 y$ that lies above the plane region with $0 \le x \le 1$ and $0 \le y \le x$.

Review Problems, VII

(Rev-#7) Find the circulation of the vector field $\mathbf{F} = \left\langle 2xy, x^2, y \right\rangle$ around the counterclockwise boundary of the portion of the surface $z = x^2 y$ that lies above the plane region with $0 \le x \le 1$ and $0 \le y \le x$.

- Use Stokes's theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$ where S is the given portion of the surface.
- We can parametrize S as $\mathbf{r}(s,t) = \left\langle s, t, s^2 t \right\rangle$ for $0 \le s \le 1$, $0 \le t \le s$, and then $\nabla \times \mathbf{F} = \langle 1-0, 0-0, 2x-2x \rangle = \langle 1, 0, 0 \rangle$ while $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 1, 0, 2st \rangle \times \left\langle 0, 1, s^2 \right\rangle = \left\langle -2st, -s^2, 1 \right\rangle$ (correct orientation since z-coordinate is positive).
- Then $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -2st$, and so the surface integral is $\displaystyle\int_0^1 \int_0^s -2st\, dt\, ds = \boxed{-1/4}$.

Review Problems, VIII

(Rev-#8) Find the flux of $\mathbf{F} = \langle 3x^2y, xy^2, 8xy \rangle$ outward through the surface S made up of the portions of the five planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 1$, with $0 \leq x, y, z \leq 1$.

Review Problems, VIII

(Rev-#8) Find the flux of $\mathbf{F} = \langle 3x^2y, xy^2, 8xy \rangle$ outward through the surface S made up of the portions of the five planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 1$, with $0 \le x, y, z \le 1$.

- We can use the divergence theorem here. However, note that the surface is not closed, so we must close it and then subtract the flux through the extra plane.
- We close it by including the plane $z = 0$ with $0 \le x, y \le 1$.
- Then the solid is $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and also $\mathrm{div}(\mathbf{F}) = 6xy + 2xy + 0 = 8xy$.
- Thus by the divergence theorem, the flux through the solid is $\iiint_D \mathrm{div}(\mathbf{F})\, dV = \int_0^1 \int_0^1 \int_0^1 8xy\, dz\, dy\, dx = 2$.

Review Problems, IX

(Rev-#8) Find the flux of $\mathbf{F} = \left\langle 3x^2y, xy^2, 8xy \right\rangle$ outward through the surface S made up of the portions of the five planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 1$, with $0 \le x, y, z \le 1$.

- For the piece being subtracted, which is the portion of the plane $z = 0$ where $0 \le x \le 1$ and $0 \le y \le 1$, we have a parametrization $\mathbf{r}(s, t) = \langle s, t, 0 \rangle$ for $0 \le s \le 1$, $0 \le t \le 1$.
- Then $\mathbf{n} = (d\mathbf{r}/ds) \times (d\mathbf{r}/dt) = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$, but this has the wrong orientation since it must point downward (that is the outward direction relative to the cube).
- Then $\mathbf{F} \cdot (-\mathbf{n}) = -8st$, and so the surface integral is $\int_0^1 \int_0^1 -8st\, dt\, ds = -2$.
- Since the flux across all six planes was 2, that means the flux across the remaining five planes is $2 - (-2) = \boxed{4}$.

Review Problems, X

(Rev-#9) Find the flux of $\mathbf{F} = \left\langle xy^2 + e^z,\ x^2y + e^{2z},\ \sqrt{x^2 + y^2} \right\rangle$ through the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane.

Review Problems, X

(Rev-#9) Find the flux of $\mathbf{F} = \left\langle xy^2 + e^z,\, x^2y + e^{2z},\, \sqrt{x^2+y^2} \right\rangle$ through the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane.

- We can use the divergence theorem here. However, note that the surface is not closed, so we must close it and then subtract the flux through the extra piece.
- We close the surface by including the bottom disc with $x^2 + y^2 \leq 1$ and $z = 0$.
- The solid is $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 1 - r^2$ in cylindrical and also $\operatorname{div}(\mathbf{F}) = y^2 + x^2 = r^2$.
- Thus by the divergence theorem, the flux through the solid is $\iiint_D \operatorname{div}(\mathbf{F})\, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^2\, r\, dz\, dr\, d\theta = \pi/6$.

Review Problems, XI

(Rev-#9) Find the flux of $\mathbf{F} = \left\langle xy^2 + e^z,\ x^2y + e^{2z},\ \sqrt{x^2 + y^2} \right\rangle$ through the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane.

- For the piece being subtracted, we have a parametrization $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ for $0 \le \theta \le 2\pi$ and $0 \le r \le 1$.
- Then $\mathbf{n} = (d\mathbf{r}/dr) \times (d\mathbf{r}/d\theta) = \langle \cos\theta, \sin\theta, 0\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle = \langle 0, 0, r\rangle$, but this has the wrong orientation since it must point downward.
- Then $\mathbf{F} \cdot (-\mathbf{n}) = -r\sqrt{x^2 + y^2} = r^2$, and so the surface integral is $\int_0^{2\pi} \int_0^1 -r^2 \, dr \, d\theta = -2\pi/3$.
- Thus, the flux across just the top portion is the difference $\pi/6 - (-2\pi/3) = \boxed{5\pi/6}$.

Review Problems, XII

(Rev-#10) Evaluate the flux of $\mathbf{F} = \left\langle x^3 + 2xz^2,\, x^2y + 2yz^2,\, 4y^2z \right\rangle$ outward through the upper half of the sphere $x^2 + y^2 + z^2 = 5$.

Review Problems, XII

(Rev-#10) Evaluate the flux of
$\mathbf{F} = \langle x^3 + 2xz^2, x^2y + 2yz^2, 4y^2z \rangle$ outward through the upper half of the sphere $x^2 + y^2 + z^2 = 5$.

- We can use the divergence theorem here. However, note that the surface is not closed, so we must close it and then subtract the flux through the extra piece.
- We close the surface by including the bottom disc with $x^2 + y^2 \leq 5$ and $z = 0$.
- The solid is $0 \leq \varphi \leq \pi/2$, $0 \leq \theta \leq 2\pi$, $0 \leq \rho \leq \sqrt{5}$ in spherical and also $\mathrm{div}(\mathbf{F}) = (3x^2+2z^2)+(x^2+2z^2)+4y^2 = 4(x^2+y^2+z^2) = 4\rho^2$.
- Thus by the divergence theorem, the flux through the solid is $\iiint_D \mathrm{div}(\mathbf{F})\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\sqrt{5}} 4\rho^2 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 40\pi\sqrt{5}$.

Review Problems, XIII

(Rev-#10) Evaluate the flux of $\mathbf{F} = \left\langle x^3 + 2xz^2,\ x^2y + 2yz^2,\ 4y^2z \right\rangle$ outward through the upper half of the sphere $x^2 + y^2 + z^2 = 5$.

- For the piece being subtracted, we have a parametrization $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ for $0 \le \theta \le 2\pi$ and $0 \le r \le \sqrt{5}$.
- Then $\mathbf{n} = (d\mathbf{r}/dr) \times (d\mathbf{r}/d\theta) = \langle \cos\theta, \sin\theta, 0\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle = \langle 0, 0, r\rangle$, but this has the wrong orientation since it must point downward.
- Then $\mathbf{F} \cdot (-\mathbf{n}) = 0$, so the flux through the bottom is simply zero (we don't even have to set up the integral).
- Hence the flux through the top piece is simply $\boxed{40\pi\sqrt{5}}$.

Final Remarks

I will have office hours on Thursday this week at the usual times, and also on Monday and Wednesday next week, times TBA.

[1] In summer-2 I am teaching Math 3081 (Probability and Statistics) and in the fall I am teaching Math 2331 (Linear Algebra), in case you have more math courses to take!

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The online format for the course isn't ideal, but I hope you found it (at least) tolerable, if not entirely passable, adequate, acceptable, okay, decent, suitable, or even satisfactory.

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This is one of my favorite calculus-level courses[1] to teach, and I hope you enjoyed the course this semester. If you did, please do make sure to fill out the TRACE evaluations.

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Happy studying, good luck on your other exams, and I will (hopefully) see you at the review session!

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Summary

We did some review problems for the final exam.

Next lecture: There isn't one, the course is over :-(

Don't forget about the final exam review session, where I will do a bunch of problems from the old finals. Please vote in the Piazza poll to select the time.