Math 2321 (Multivariable Calculus)

Lecture #36 of 37 $\sim$ April 19, 2021

Applications of Multivariable Calculus, Part 2

- The Heat and Wave Equations
- Modeling With Partial Differential Equations
- Maxwell's Equations and Electromagnetism

This material represents §4.7.2-4.7.4 from the course notes. (This material is for fun only!)

Logistical Things

- On Wednesday I will do some problems from the “extra” review sheet on Stokes’s theorem and the divergence theorem.
- I have applied all of the remaining WeBWorK extensions to set 12 for everyone.
- The final exam is 10:30am-12:30pm on Thursday, April 29th, unless you have already made alternate arrangements due to an exam conflict.
- The best way to study for our final is to practice with the old finals. I will run a 2-hour review session on Monday, Tuesday, or Wednesday next week, date + time decided by Piazza vote, where I will do a bunch of old final exam problems.
- Your course grade is the maximum of
  16% WeBWorK + $3 \times 18\%$ each midterm + 30% final, and
  20% WeBWorK + $2 \times 20\%$ top two midterms + 40% final.

The Heat Equation, I

One place that many of the ideas in this course show up is in modeling physical phenomena. One famous example is the heat equation $f_t = \gamma \nabla \cdot (\nabla f)$, where $f(x, y, z, t)$ gives the temperature of an object at a point (x, y, z) at time t and γ is a rate constant.

- In standard notation, the heat equation reads as $f_t = \gamma(f_{xx} + f_{yy} + f_{zz})$.
- For shorthand, even though it is technically bad notation, we often write the operator $\nabla \cdot \nabla$ as $\nabla^2 = \langle \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \rangle$. This operator is called the Laplacian and is often also written as Δ.
- The heat equation is essentially a rephrasing of the second law of thermodynamics and Newton's law of cooling (heat flows from hot things to cold ones at a rate proportional to the difference in temperatures).

The Heat Equation, II

In fact, we can actually derive the heat equation $f_t = \gamma \nabla^2 f$ using the divergence theorem, as follows:

- Let $H(t)$ be the amount of heat contained in a region D. Then $H(t) = \iiint_D \alpha f(x, y, z, t)\, dV$ since temperature is a measure of heat density.
- Taking the derivative with respect to t of both sides yields $H_t(t) = \iiint_D \alpha f_t(x, y, z, t)\, dV$ (this is essentially a combination of the mixed-partials theorem and Fubini's theorem to move the t-derivative inside the integral).

The Heat Equation, III

The heat flow into the solid D is $H_t(t) = \iiint_D \alpha f_t(x, y, z, t)\, dV$.

- The heat flow $H_t(t)$ is also given by computing the flux of the heat flowing through the boundary of the surface. The vector field modeling the heat flow is ∇f, so the flux of this field is the surface integral $\iint_S \beta(\nabla f) \cdot \mathbf{n}\, d\sigma$.
- By the divergence theorem, the surface integral equals $\iiint_D \beta\nabla \cdot (\nabla f)\, dV = \iiint_D \beta\nabla^2 f\, dV$.
- Since we have an equality $\iiint_D \alpha f_t\, dV = \iiint_D \beta\nabla^2 f\, dV$ on every solid D, that means the underlying functions αf_t and $\beta\nabla^2 f$ are equal everywhere.
- Moving the constant factors around then yields $f_t = \gamma\nabla^2 f$: this is the heat equation.

The Heat Equation, IV

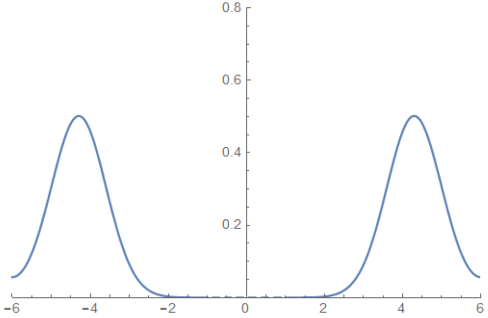
The heat equation $f_t = \gamma \nabla^2 f$ also shows up in many other places.

- In probability theory, the heat equation shows up as a very natural continuous model for random walks. In physics, this is closely connected with the study of Brownian motion.
- In financial mathematics, the Black-Scholes equation (which is used for computing the proper price of options) is a minor variation of the heat equation: if V is the price of an option as a function of the asset S and time t, then it says $V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$. Up to the coefficients and the first-order and constant terms, it is essentially $V_t = \gamma V_{SS}$, which is a one-dimensional heat equation.
- In quantum mechanics, Schrödinger's equation reads as $H|\psi(t)\rangle = i\bar{h}\frac{\partial}{\partial t}|\psi(t)\rangle$. For a single particle, H is (essentially) the Laplacian operator ∇^2, so this is (very roughly!) a heat equation with an imaginary constant factor.

The Wave Equation, I

Another famous partial differential equation is the <u>wave equation</u>: $f_{tt} = \gamma \nabla^2 f$, where $f(x, y, z, t)$ measures the intensity of a wave at a point (x, y, z) in space at time t.

- In standard notation, the wave equation reads as $f_{tt} = \gamma(f_{xx} + f_{yy} + f_{zz})$.



The Wave Equation, II

Pleasantly enough, the one-dimensional wave equation $f_{tt} = c^2 f_{xx}$ can actually be solved more or less explicitly.

- If we write $a = x - ct$, $b = x + ct$, then by the chain rule, the wave equation is equivalent to $f_{ab} = 0$, which has the simple solution $f(a, b) = F(a) + G(b)$ for arbitrary functions F and G (simply antidifferentiate twice).
- This yields a general solution $f(x, t) = F(x - ct) + G(x + ct)$.
- This is the sum of a "left-moving function" $F(x - ct)$ and a "right-moving function" $G(x + ct)$ as t increases.
- Imagine plucking a string on an instrument and you will have exactly the right idea!

The Wave Equation, III

Like the heat equation, the wave equation can also be derived from basic physical principles using the divergence theorem.

- Specifically, suppose D is any region. Then the acceleration within D is the second t-derivative of $\iiint_D f\,dV$, which is $\iiint_D f_{tt}\,dV$.
- The vector field $\mathbf{F}$ modeling the force imparted by the wave is ∇f, and so the total force acting on D through its boundary S is equal to the surface integral $\iint_S (\nabla f)\cdot \mathbf{n}\,d\sigma$, which equals $\iiint_D \nabla^2 f\,dV$ by the divergence theorem.
- Applying Newton's second law ($F = ma$) and equating the two triple integrals on every D then gives the wave equation: $f_{tt} = \gamma \nabla^2 f$.

Modeling, I

In most cases, the differential equation (or equations) modeling a physical phenomenon are difficult if not impossible to solve exactly.

- There are many methods for finding approximate solutions.
- One approach is to employ a "step method" and linearization: we take a linearization of the system and then move a small step forward in time (the idea being that for a small step, the linearization is a good approximation of the original).
- We then iterate this procedure with the new system that has been moved forward: we linearize and then move a small step forward in time, repeatedly.
- Techniques like this one can be used to analyze models for weather and climate, urban planning, epidemiology (e.g., during global pandemics), ecology, experimental biology, chemistry, and physics, and just about everywhere else....

Modeling, II

I'll also mention a related idea involving numerical methods.

- In many applications, one needs to search for a minimum or maximum value of some function.
- For example, if one wants to model a chemical reaction computationally (which is now possible to do with modern supercomputers), one needs to compute minimum-energy configurations of molecules.
- To perform such simulations, the computer must use step methods to iterate each interaction of particles in small time intervals, and search for the minimum-energy state.
- To find such a state, one may use a "gradient-step method": compute the current energy, and then step in the opposite direction of the gradient of this energy function.

Modeling, III

As we have discussed, the gradient points in the direction of maximum increase of a function, so at each stage, the search will move in the direction that lowers total energy.

- Eventually, a gradient-step algorithm will reach a state in which the gradient is zero, which is a critical point of the energy function.
- To determine whether the energy is actually minimized then requires classifying the resulting critical point as a local minimum, local maximum, or saddle point.
- Of course, in actual practice, the search space is much larger than the 2-dimensional examples we treated in this class (typically it has hundreds or thousands of variables).
- But the general principle, that one may classify the type of critical point by using a "second derivatives test", turns out to be very similar.

Modeling, IV

In many applications, we have a model that we want to fit to a given data set.

- In statistics[1] there are various methods for making "parameter estimates" of this type: indeed, a major component of statistics is about developing methods for making parameter estimates from data seta.
- A computationally convenient technique, frequently used in practice, is to employ a least-squares regression: we minimize the sum of the squared errors between the predicted and observed values.
- The reason to use the sum of squares, rather than something else like the sum of the absolute errors, is because we can minimize the resulting function using calculus.

[1] To learn more, take Math 3081. (I am teaching it in summer 2!)

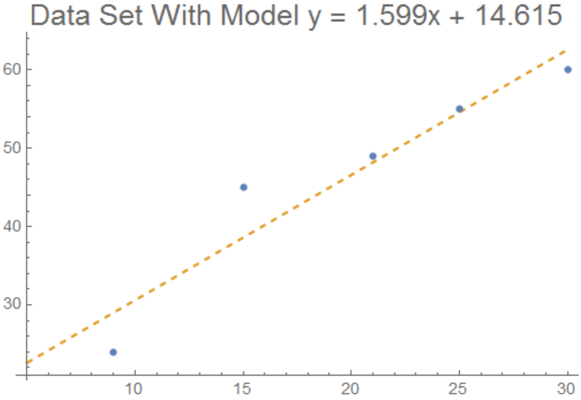
Modeling, V

Perhaps the simplest example is to fit a linear function $y = ax + b$ to a data set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$.

- The function to minimize for the linear model above is $E(a, b) = (ax_1 + b - y_1)^2 + (ax_2 + b - y_2)^2 + \cdots + (ax_n + b - y_n)^2$.
- To minimize this function we set the two partial derivatives $\partial E/\partial a$ and $\partial E/\partial b$ equal to zero.
- We have $\partial E/\partial a = 2x_1(ax_1 + b - y_1) + \cdots + 2x_n(ax_n + b - y_n)$ and $\partial E/\partial b = 2(ax_1 + b - y_1) + \cdots + 2(ax_n + b - y_n)$, so that $a \sum x_i^2 + b \sum x_i = \sum x_i y_i$ and $a \sum x_i + nb = \sum y_i$.
- This is a linear system for a and b, with solution $a = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$ and $b = \frac{1}{n}(\sum y_i - a \sum x_i)$.
- These two values a and b together give the equation for the famous least-squares regression line to a data set.

Modeling, VI

Here, for example, is the plot of a data set $\{(9, 24), (15, 45), (21, 49), (25, 55), (30, 60)\}$ along with its least-squares regression line $y = 1.599x + 14.615$:



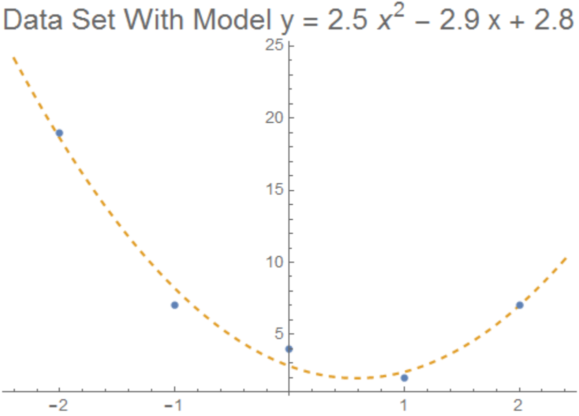
Modeling, VII

However, the method of least squares is quite robust, and we can use it for all sorts of other models too.

- For example, we can use more complicated functions than mere lines – e.g., we could try to fit a quadratic function $y = ax^2 + bx + c$ to a data set.
- The procedure is essentially the same as before: we write down the sum of squared errors and then minimize it using calculus, by setting all of the partial derivatives equal to zero.
- Here, the function is $E(a, b, c) = (ax_1^2 + bx_1 + c - y_1)^2 + \cdots + (ax_n^2 + bx_n + c - y_n)^2$.
- We then calculate $\partial E/\partial a$, $\partial E/\partial b$, and $\partial E/\partial c$ and set them equal to zero. The resulting system will be linear in a, b, c and we can then solve it to compute the predicted coefficients a, b, c.

Modeling, VIII

Here, for example, is the plot of a data set $\{(-2, 19), (-1, 7), (0, 4), (1, 2), (2, 7)\}$ along with the parabola $y = -2.5x^2 - 2.9x + 2.8$ of best fit:



Maxwell's Equations and Electromagnetism, I

We now briefly discuss Maxwell's equations of electromagnetism:

- Here, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is electric charge density, and ϵ_0 and μ_0 are constants. (We assume no current $\mathbf{J}$ here.)

Law	Integral Form	Differential Form
Gauss (E)	$\displaystyle\oiint_S \mathbf{E}\cdot\mathbf{n}\,d\sigma = \frac{1}{\epsilon_0}\iiint_D \rho\,dV$	$\displaystyle\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$
Gauss (M)	$\displaystyle\oiint_S \mathbf{B}\cdot\mathbf{n}\,d\sigma = 0$	$\nabla\cdot\mathbf{B} = 0$
Maxwell-Faraday	$\displaystyle\oint_C \mathbf{E}\cdot\mathbf{T}\,ds = -\frac{d}{dt}\left[\iint_\Sigma \mathbf{B}\cdot\mathbf{n}\,d\sigma\right]$	$\displaystyle\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
Ampère	$\displaystyle\oint_C \mathbf{B}\cdot\mathbf{T}\,ds = \mu_0\epsilon_0\frac{d}{dt}\left[\iint_\Sigma \mathbf{E}\cdot\mathbf{n}\,d\sigma\right]$	$\displaystyle\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$

Maxwell's Equations and Electromagnetism, II

In the two Gauss laws, S is a closed surface enclosing the solid region D, so if we apply the divergence theorem, we may convert the surface integral into a triple integral.

Law	Integral Form	Differential Form
Gauss (E)	$\oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \frac{1}{\epsilon_0} \iiint_D \rho\, dV$	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$
Gauss (M)	$\oiint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma = 0$	$\nabla \cdot \mathbf{B} = 0$

- For the electric field law, by the divergence theorem we have $\oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \iiint_D (\nabla \cdot \mathbf{E})\, dV$, so the integral form is equivalent to saying $\iiint_D (\nabla \cdot \mathbf{E})\, dV = \frac{1}{\epsilon_0} \iiint_D \rho\, dV$.
- This equality holds on every solid region D, so the integrands $\nabla \cdot \mathbf{E}$ and ρ/ϵ_0 are equal: this is the differential form.
- A similar argument works for the magnetic field law.

Maxwell's Equations and Electromagnetism, III

In the other two laws, Σ is a surface with counterclockwise boundary curve C, so we can apply Stokes's theorem.

Law	Integral Form	Differential Form
Maxwell-Faraday	$\oint_C \mathbf{E} \cdot \mathbf{T}\, ds = -\frac{d}{dt}\left[\iint_\Sigma \mathbf{B} \cdot \mathbf{n}\, d\sigma\right]$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
Ampère	$\oint_C \mathbf{B} \cdot \mathbf{T}\, ds = \mu_0 \epsilon_0 \frac{d}{dt}\left[\iint_\Sigma \mathbf{E} \cdot \mathbf{n}\, d\sigma\right]$	$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

- For the Maxwell-Faraday law, by Stokes' theorem the integral $\oint_C \mathbf{E} \cdot \mathbf{T}\, ds$ equals $\iint_\Sigma (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, d\sigma$.
- Thus, the integral form is equivalent to $\iint_\Sigma (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, d\sigma = -\frac{d}{dt} \iint_\Sigma \mathbf{B} \cdot \mathbf{n}\, d\sigma = -\iint_\Sigma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, d\sigma$.
- Since this holds on every surface, the two fields $\nabla \times \mathbf{E}$ and $-\frac{\partial \mathbf{B}}{\partial t}$ must be equal. This is the differential form.
- A similar argument yields the two versions of Ampère's law.

Maxwell's Equations and Electromagnetism, IV

We can actually derive Gauss's law for both electricity and magnetism as a consequence of more general properties of inverse-square laws.

- Coulomb's law says that the electric force between two particles is proportional to each of their charges and inversely proportional to the square of the distance between them. (Compare to Newton's law of gravitation.)
- More explicitly, for a single point charge q at the origin, the electric field $\mathbf{E}$ equals $\mathbf{E}(\mathbf{r}) = \dfrac{q}{4\pi\epsilon_0}\dfrac{\mathbf{r}}{||\mathbf{r}||^2}$.

Maxwell's Equations and Electromagnetism, V

More explicitly, for a single point charge q at the origin, the electric field $\mathbf{E}$ equals $\mathbf{E}(\mathbf{r}) = \dfrac{q}{4\pi\epsilon_0}\dfrac{\mathbf{r}}{||\mathbf{r}||^2}$.

- We can then compute the surface integral through the sphere of radius a centered at the origin directly: the normal vector $\mathbf{n} = \mathbf{r}$, and so the surface integral in spherical coordinates is $\displaystyle\int_0^{2\pi}\int_0^{\pi}\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{a^2}\cdot\mathbf{r}\,d\varphi d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{q}{4\pi\epsilon_0}\,d\varphi d\theta = \frac{q}{\epsilon_0}$, because the dot product $\mathbf{r}\cdot\mathbf{r} = a^2$ on the sphere of radius a.
- This agrees with the triple integral of Gauss's law for the case of a single particle (the triple integral is simply $\dfrac{q}{\epsilon_0}$).

This may seem like a very special case of Gauss's law, but we can actually use it to get the general version.

Maxwell's Equations and Electromagnetism, VI

First, we extend Gauss's law to arbitrary surfaces.

- If we have an arbitrary closed surface T containing the origin, choose a sphere S that encloses it and take D to be the region between the two surfaces.
- Then, by the divergence theorem, we see that $\iiint_D (\nabla \cdot \mathbf{E})\, dV = \oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma - \oiint_T \mathbf{E} \cdot \mathbf{n}\, d\sigma$ (the minus sign is because the normal vector for T points inward).
- But for $\mathbf{E}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \dfrac{q}{4\pi\epsilon_0} \dfrac{\langle x, y, z \rangle\rangle}{(x^2 + y^2 + z^2)^{3/2}}$, we can compute explicitly that $\nabla \cdot \mathbf{E} = 0$ for $(x, y, z) \neq (0, 0, 0)$.
- Thus, since D does not contain the origin, the triple integral is zero, and so we conclude that $\oiint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \oiint_T \mathbf{E} \cdot \mathbf{n}\, d\sigma$.
- This means that the Gauss law result holds for a single particle and an arbitrary surface T.

Maxwell's Equations and Electromagnetism, VII

Finally, we can use the fact that Gauss's law holds for a single particle and an arbitrary surface to obtain the result for arbitrary charge distributions and arbitrary surfaces.

- The idea is simply to sum over all of the various charges, and observe that both the surface integral and the triple integral are consistent with summing over charges.
- Then we may take a limit of finite sums of charges to obtain the result for arbitrary charge distributions.
- This establishes Gauss's law, as claimed.

For the Gauss law for magnetic fields $\oiint_S \mathbf{B} \cdot \mathbf{n}\, d\sigma = 0$, the result is quite a bit simpler.

- The point is that there is no magnetic equivalent of charge (this would be a "magnetic monopole", of which no experimental observation has ever been made), and so the resulting triple integral of "magnetic charge" is simply zero.

Maxwell's Equations and Electromagnetism, VIII

Law	Differential Form
Gauss (E)	$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$
Gauss (M)	$\nabla \cdot \mathbf{B} = 0$
Maxwell-Faraday	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
Ampère	$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$

- Both **E** and **B** have 3 components. The two Gauss's laws each impose one condition on the components, while the other two laws each impose three conditions. So we seemingly have 8 conditions on the 6 components.

- But in fact, there are two redundant conditions, which are accounted for by the div-curl identity, which says $\mathrm{div}(\mathrm{curl}(\mathbf{F})) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field **F**.
- So in fact, there are six conditions on the six components, which is "exactly enough" to determine them.

Maxwell's Equations and Electromagnetism, IX

There is a slick way to prove the div-curl identity $\mathrm{div}(\mathrm{curl}(\mathbf{F})) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$ using the divergence theorem and Stokes's theorem.

- First, if $\mathbf{F}$ is any vector field and S is any closed surface, we claim that the flux of the curl $\iint_S \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma$ is zero.
- To see this, draw any closed curve C on the surface that cuts it into two pieces, and apply Stokes's theorem: the flux across one piece will be the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, while the flux across the other piece will be the negative $-\oint_C \mathbf{F} \cdot d\mathbf{r}$, since the boundaries of these two surfaces are both C, but with opposite orientations.
- So $\iint_S \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0$. Now apply the divergence theorem: if D is the interior of S, then we get $\iiint_D \mathrm{div}(\mathrm{curl}(\mathbf{F}))\, dV = 0$. But this holds for every possible region D, so the integrand itself must be zero everywhere.

Maxwell's Equations and Electromagnetism, X

One more bit of fun with Maxwell's equations:

- Suppose that the charge density q is zero everywhere: then Gauss's law for the electric field says that $\mathrm{div}(\mathbf{E}) = 0$.
- It is not hard to verify the "curl-curl" identity $\mathrm{curl}(\mathrm{curl}(\mathbf{F})) = \mathrm{grad}(\mathrm{div}(\mathbf{F})) - \nabla^2 \cdot \mathbf{F}$ just by writing it out.
- Applying this to the vector field $\mathbf{E}$ yields $\nabla \times (\nabla \times \mathbf{E}) = \mathrm{grad}(\mathrm{div}(\mathbf{E})) - \nabla^2 \cdot \mathbf{E} = -\nabla^2 \cdot \mathbf{E}$ since $\mathrm{div}(\mathbf{E}) = 0$.
- Also, by the other Maxwell's equations, $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ so

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}[\nabla \times \mathbf{B}] = \frac{\partial}{\partial t}\left[\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right] = \mu_0 \epsilon_0 \mathbf{E}_{tt}.$$

- So, this vector field identity tells us $-\nabla^2 \cdot \mathbf{E} = -\mu_0 \epsilon_0 \mathbf{E}_{tt}$: thus, $\mathbf{E}$ satisfies the wave equation! (Likewise for $\mathbf{B}$.)

Maxwell's Equations and Electromagnetism, XI

Since $\mu_0 \epsilon_0 = c^2$, the calculation from the last slide tells us that in the absence of charge, **E** and **B** both satisfy the wave equation $\nabla^2 \cdot \mathbf{E} = c^2 \mathbf{E}_{tt}$ with speed parameter c (the speed of light).

- We see, therefore, that analysis of Maxwell's equations leads (more or less directly) to a derivation of the phenomenon of electromagnetic waves.
- Of course, electromagnetic waves are a quite well-understood concept in the 21st century, so it is likely not very surprising to you that electromagnetic waves exist.
- But Maxwell published his original papers detailing these equations, and deducing some of these consequences that unify electricity and magnetism, in 1861.
- All of this analysis was done by hand, and it provided the theoretical foundation for the development of all of this wonderful technology we now take for granted.

Summary

We discussed some basic facts about the heat and wave equations.

We discussed some results about modeling, approximation, and least-squares estimation.

We discussed Maxwell's equations and electromagnetism.

Next lecture: Final exam review, part 1.